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Abstract

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SOME REMARKS ON TRANSFINITE DIMENSION

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In Sec. A it is proved that every infinite-dimensional space is either itself strongly infinite-dimensional, or else the complement to one of its points is strongly infinite-dimensional. In Sec. B the structure of metric spaces having large transfinite dimension is investigated (^{2}). In Sec. C, for the case of spaces possessing a base that decomposes into the sum of a countable number of star-finite covers, a theorem of Hurewicz (^{6}) is generalized, and in Sec. D, to the case of arbitrary complete metric spaces, a theorem of E. Sklyarenko (^{3}) is extended.

A. For noncompact spaces I proposed the following definition (see (^{1})):

A. A space is **weakly infinite-dimensional** if, for every countable system γ of pairs of closed sets A_i, B_i such that $A_i \cap B_i = \emptyset$, one can choose partitions* C_i between A_i and B_i, a finite number of which already have empty intersection:

$$\bigcap_{i<N} C_i = \emptyset, \quad N = N(\gamma).$$

It is convenient because, by means of the following device of Sklyarenko, the study of weakly infinite-dimensional noncompact spaces is reduced to the compact case. We shall call a **Sklyarenko representation** any representation of a space R as the sum of a countable number of finite-dimensional** open sets Γ_n and of a compact closed set Φ, such that

$$R \setminus \Phi = \bigcup \Gamma_n,$$

and such that every sequence of points x_i, taken from the sum $\bigcup \Gamma_n$, which has no limit points in R, is wholly contained in one of the summands Γ_n, possibly with the exception of a finite number of terms x_i.

Sklyarenko' s device. *A metric space is weakly infinite-dimensional if and only if it has such a Sklyarenko representation in which the compactum Φ is weakly infinite-dimensional* (see (^{1}), Theorem 3).

If the compactum Φ is empty, then the number of nonempty summands Γ_n is finite. Therefore the sum of pairwise disjoint open sets whose dimensions increase is no longer weakly infinite-dimensional, or, what is the same thing, is strongly infinite-dimensional. This fact and the following theorem show that, in a certain sense, there are too many strongly infinite-dimensional spaces.

Theorem 1. *A metric space is infinite-dimensional if and only if either it is itself strongly infinite-dimensional, or the complement to one of its points is strongly infinite-dimensional***.

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* A closed set C of a space R is called a **partition** between the sets A and B if the complement $R \setminus C$ is split into two disjoint open sets G and H such that $A \subseteq G$ and $B \subseteq H$.

** By dimension we shall everywhere mean the dimension dim, defined with the aid of covers. For metric spaces (which alone we shall consider) the dimension dim is equal to the large inductive dimension Ind, defined by induction over closed sets (see ([8,9])).

*** The conditions of the theorem need not exclude each other: the Hilbert cube and the complement of this cube to any of its points are strongly infinite-dimensional.

Proof. Sufficiency is clear. The necessity of the theorem follows from the fact that in every infinite-dimensional but weakly infinite-dimensional space R there exist points all of whose neighborhoods are infinite-dimensional (see ([1]), proof of Theorem 3), since the following is true:

Theorem 1′. *In a metric space R, the complement $R \setminus x$ of any point x all of whose neighborhoods are infinite-dimensional is strongly infinite-dimensional.*

Proof. Let $U_n x$ be the spherical neighborhoods of the point x of "radius" $1/n$. It is easy to see that in our case there is a sequence n_i such that for the differences

$$F_n = [U_n x] \setminus U_{n+1} x$$

we have $\dim F_{n_i} \geq i$. Therefore the sum $\bigcup F_{n_i}$, and with it the complement $R \setminus x$, are strongly infinite-dimensional. Theorems 1′ and 1 are proved.

B. The study of large transfinite dimension was begun in ([2]). However, the following proposition, easily proved by induction*, was not noticed by me:

Theorem 2. *Every metric space possessing large transfinite dimension* Ind *is weakly infinite-dimensional.*

Following Nagata, let us say that a space is countable-dimensional if it is the sum of a countable number of zero-dimensional (in the sense of dimension dim!) sets.

Theorem 2′. *Every metric space possessing large transfinite dimension is countable-dimensional.*

Proof. By Theorem 2 and Sklyarenko' s theorem, a metric space R possessing dimension Ind has a representation

$$R = \bigcup \Gamma_n \cup \Phi,$$

in which the compactum Φ has dimension Ind. Since each finite-dimensional set Γ_n is countable-dimensional (see (^{8},^{9})) and every compactum possessing dimension Ind is also countable-dimensional (see (^{2}), Theorem 4), R is countable-dimensional as well.

For what follows define the function** $\beta(\alpha)$, beginning with the value $\beta(-1) = \omega_0$, by the following recurrence relations:

$$\beta(\alpha) = \sup_{\alpha' < \alpha} \beta(\alpha') + 1.$$

It is easy to show that: 1) if $\alpha < \alpha'$, then $\beta(\alpha) < \beta(\alpha')$; 2) if $\alpha < \omega_1$, then also $\beta(\alpha) < \omega_1$; 3) $\alpha \leq \beta(\alpha)$ for any α.

The following theorem is proved by induction:

Theorem 3. *Let a metric space R have a Sklyarenko representation*

$$R \setminus \Phi = \bigcup \Gamma_n,$$

in which the compactum Φ has dimension $\operatorname{Ind}\Phi$; then the space R also has dimension $\operatorname{Ind} R$, and moreover

$$\operatorname{Ind} R \leq \beta(\operatorname{Ind}\Phi).$$

Corollary 1. *A metric space has dimension* Ind *if and only if it has a Sklyarenko representation*

$$R \setminus \Phi = \bigcup \Gamma_n,$$

in which the compactum Φ is countable-dimensional (has transfinite dimension).

Corollary 2. *The values of the large transfinite dimension for metric spaces are less than ω_1.*

For the derivation of the corollaries see (^{2}), p. 193.

C. It is easy to show that for spaces with a countable base in Theorem 2$'$ the dimension Ind can be replaced by the small transfinite dimension ind (see (^{6}), p. 75). It turns out that this can be done under considerably broader assumptions.

A covering γ is called **star-finite** if each of its elements intersects only a finite number of other elements of the same

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[*] Noting that for any closed sets A and B of a metric space R, every partition C' in a subspace R' between the sets $A \cap R'$ and $B \cap R'$ can be extended (by virtue of hereditary normality) to a partition C between A and B so that $C \cap R' = C'$.

[**] The argument α and the value $\beta(\alpha)$ are ordinal numbers.

coverings. We shall call a space R **strongly metrizable** if it has a base decomposable into the sum of a countable number of star-finite coverings*. Every strongly paracompact (^{4}) metrizable space is strongly metrizable. However, there exist strongly metrizable spaces that cannot even be represented as the sum of a countable number of closed strongly paracompact spaces (see (^{11})).

Theorem 4. *If a metrizable space that is the sum of a countable number of strongly metrizable sets has small transfinite dimension* ind*, then it is countably dimensional.*

Lemma 1∗∗. *If a strongly metrizable space R has small dimension* $\operatorname{ind} R = \alpha$, *then in every covering of it one can inscribe a covering $\{U_\lambda\}$, decomposable into the sum of a countable number of discrete*** systems, in which the boundaries of all elements U_λ have small dimension $< \alpha$.*

Proof. Proving the theorem by induction, we find that in Lemma 1 the boundaries of all elements U_λ may be assumed countably dimensional. It follows that if a strongly metrizable space R has small dimension ind, then it also has a base ω, decomposable into the sum of a countable number of locally finite coverings, such that the boundary of each of its elements is countably dimensional. Therefore the sum of the boundaries Σ of all elements of the base ω will also be countably dimensional. One can prove that the set $R \setminus \Sigma$ has a base decomposable into the sum of a countable number of coverings of multiplicity 1. Hence it follows that $\dim(R \setminus \Sigma) = 0$ (see (^{4}) or (^{8})). Therefore R is also countably dimensional.

G. The assertion converse to Theorem 4 is easy to prove for arbitrary complete metric spaces (see (^{6}), p. 75). The condition of completeness (in the sense of Čech) is essential even for spaces with a countable base. The following theorem substantially covers this converse assertion and, at the same time, the theorem of E. G. Sklyarenko from (^{3}).

Theorem 5. *Every complete metric space that is the image of a countably dimensional space under a closed countable-to-one mapping has small inductive dimension.*

Proof. By Nagata' s theorem (^{7}), every countably dimensional space is the image of a zero-dimensional space under a closed finite-to-one mapping. Therefore it suffices to show that no closed mapping f of a zero-dimensional space X onto a complete metric space Y not having small dimension ind is countable-to-one. By Stone' s theorem (^{10}), the boundaries of the complete inverse images $f^{-1}y$ are bicompact. The set H, which is the sum of the nonempty boundaries of all inverse images $f^{-1}y$ and of points x, chosen one from each open-and-closed inverse image $f^{-1}y$, is closed and $f(H) = Y$. By Sklyarenko' s method from (^{3}), for any n one can construct open-and-closed sets $H_{i_1 \dots i_n}$ of the space H, where $i_j = 0, 1$, such that $H_{i_1 \dots i_n 0} \cap H_{i_1 \dots i_n 1} = \emptyset$, $H_{i_1 \dots i_n 0} \cup H_{i_1 \dots i_n 1} = H_{i_1 \dots i_n}$, and such that $C_n = \bigcap f H_{i_1 \dots i_k} \neq \emptyset$, where the intersection is taken over all indices $i_1 \dots i_k$, for $k \leq n$. One may require that the diameters of the sets C_n tend to zero. Then $\bigcap C_n \neq \emptyset$, and for the point $y = \bigcap C_n$ the inverse image $f^{-1} y$ has the cardinality of the continuum, as was required to prove.

Corollary. *For strongly metrizable spaces (and their countable sums) that have a complete metric, the following properties are equivalent:*

a) *the space R has small dimension* ind*;*

b) *the space R is countably dimensional;*

c) *the space R is the image of a zero-dimensional space under a closed finite-to-one mapping**.*

* Cf. the metrizability condition in (5). Spaces of this kind were apparently first considered by K. Morita.

** This lemma is a modification of the corresponding lemma of A. Zarelua (11).

*** A system of sets is called **discrete** if every point has a neighborhood meeting no more than one element of the given system.

**** Properties b) and c) are equivalent for arbitrary metric spaces (7).

c) the space R is the image of a countably dimensional space under a closed and finite-to-one mapping.

The following question is of interest:

Is Theorem 4 true for arbitrary metric spaces?

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REFERENCES

[1] E. G. Sklyarenko, *Izv. AN SSSR, Ser. Matem.*, **23**, No. 2, 197 (1959).
[2] Yu. M. Smirnov, *Izv. AN SSSR, Ser. Matem.*, **23**, No. 2, 185 (1959).
[3] E. G. Sklyarenko, *DAN*, **126**, No. 6, 1203 (1959).
[4] Yu. M. Smirnov, *Izv. AN SSSR, Ser. Matem.*, **20**, No. 2, 253 (1956).
[5] Yu. M. Smirnov, *UMN*, **6**, No. 6, 977 (1951).
[6] V. Gurevich, G. Wallman, *Dimension Theory*, Moscow, 1948.
[7] Ju. Nagata, *Proc. Japan Acad.*, **34**, No. 3, 146 (1958).
[8] K. Morita, *Math. Ann.*, **128**, 350 (1954).
[9] M. Katětov, *DAN*, **79**, No. 1, 189 (1951).

[10] A. H. Stone, *Proc. Am. Math. Soc.*, **7**, No. 4, 690 (1956).
[11] A. Zarelua, *DAN*, **141**, No. 4 (1961).

Note: Figure translations are in progress. See original paper for figures.