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Abstract

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ON A BOUNDARY-VALUE PROBLEM FOR THE EQUATION $\Delta U = U^2$

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In the present note we consider the boundary-value problem

$$\Delta U = U^2, \qquad U\big|_\Gamma = \varphi(s) \tag{1}$$

for twice continuously differentiable real functions $U(x, y)$ in a bounded domain G with boundary Γ, under continuous boundary values $\varphi(s)$.

First, the existence of a nontrivial solution of problem (1) is established for the case of zero boundary values. Then a theorem is proved on the existence of two distinct solutions of the boundary-value problem (1) under certain conditions on the function $\varphi(s)$. Further, the boundary-value problem (1) is investigated for the case of nonpositive values of the function $\varphi(s)$. In this case some sufficient conditions for the existence or nonexistence of solutions of the boundary-value problem (1) are indicated.

Let $K(P, Q)$ be the Green' s function of the operator $L(U) = \Delta U - qU$, where the function $q(x, y)$ has continuous first-order derivatives; P and Q are points of the domain G, and suppose that the equation

$$\psi(P) = \lambda \int_G K(P, Q)\psi(Q)\, dQ$$

has only positive eigenvalues.

Theorem 1. *There exists a* μ *such that the integral equation*

$$\mu V(P) = \int_G K(P, Q)V^2(Q)\, dQ \tag{2}$$

has a nontrivial solution.

Proof. Consider the auxiliary variational problem: to find the greatest value of the functional

$$F(x_1, x_2, \dots, x_n, \dots) = \frac{1}{3} \int_G \left(\sum_1^\infty x_k \varphi_k(Q) \right)^3 dQ$$

under the condition

$$H(x_1, x_2, \dots, x_n, \dots) = \frac{1}{2} \sum_1^\infty \lambda_k x_k^2 = c^2,$$

where $\varphi_k(Q)$ and λ_k are, respectively, the orthonormal eigenfunctions and eigenvalues of the kernel $K(P, Q)$. From the theorems of M. Vainberg (^{2}) on the properties of quadratic integral forms in the spaces L_q ($q \leq 2$), it follows that the functional F is defined on the manifold $H = c^2$ and is bounded in absolute value.

To prove the existence of a solution of this variational problem we apply the Ritz method.

The maximizing sequence of elements $X^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)})$ is found from the condition that the functional F_n, defined by the equality

$$F_n(x_1, x_2, \dots, x_n) = F(x_1 x_2, \dots, x_n, 0, \dots)$$

attains at the point $X^{(n)}$ its maximum d_n under the condition

$$H_n(x_1, x_2, \dots, x_n) = H(x_1, x_2, \dots, x_n, 0, \dots) = c^2.$$

The coordinates of the point $X^{(n)}$ satisfy the system of equations

$$\mu_n \lambda_i x_i^{(n)} = \int_G \left(\sum_1^n x_k^{(n)} \varphi_k(Q) \right)^2 \varphi_i(Q)\, dQ, \tag{3}$$

where

$$\mu_n = \frac{3}{2c^2} d_n.$$

By a known device, from the sequence of elements

$$\widetilde{X}^{(n)} = \left(\sqrt{\lambda_1} x_1^{(n)}, \sqrt{\lambda_2} x_2^{(n)}, \dots, \sqrt{\lambda_n} x_n^{(n)} \right)$$

one extracts a subsequence $\{\widetilde{X}^{(n_k)}\}$ weakly convergent in l_2, such that for every i

$$\sqrt{\lambda_i}\, x_i^{(n_k)} \to \sqrt{\lambda_i}\, x_i^{(0)} \quad \text{as } k \to \infty.$$

Applying the theorems of Vainberg (^{2}), from system (3) we obtain the system of equations

$$\mu \lambda_i x_i^{(0)} = \int_G \left(\sum_1^\infty x_k^{(0)} \varphi_k(Q) \right)^2 \varphi_i(Q)\, dQ;$$

the element $X^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}, \dots)$ also solves the original variational problem. For the quantity μ the inequality

$$\frac{\sqrt{2}\, c}{\lambda_1^{3/2}} \|\varphi_1\|_{L_s^3} \leq \mu \leq \frac{(\sqrt{2}c)^{3/2}}{3} \|K\|_{L_s^{3/2}(G \times G)}.$$

In obtaining the lower bound for μ it was assumed that $\varphi_1(P) \geq 0$. This is ensured if $q(x, y) \geq 0$, or if $|q(x, y)|$ is sufficiently small.

The sequence of functions

$$V_m(P) = \sum_1^m x_k^{(0)} \varphi_k(P)$$

converges uniformly in the domain G to the function $V(P)$. The function $V(P)$ is not identically zero and satisfies equation (2).

Remark. The function $U(P) = \frac{1}{\mu} V(P)$ is not identically zero and satisfies the equation

$$U(P) = \int_G K(P, Q) U^2(Q)\, dQ.$$

Theorem 2. *The boundary-value problem* (1) *with* $\varphi \equiv 0$ *has a nontrivial solution.*

Proof. It suffices to set the function $q(x, y)$ equal to zero and use the remark to Theorem 1.

Let $K_0(P, Q)$ be the Green' s function of the Laplace operator for the domain G. Denote

$$\max_G \int_G K_0(P, Q)\, dQ = K_1, \qquad \max_\Gamma |\varphi(s)| = B.$$

Theorem 3. If the function $\varphi(s)$ satisfies one of the conditions:
a) $\varphi(s) \geqslant 0$; b) $4BK_1 < 1$, then the boundary-value problem (1) has two distinct solutions.

Proof. The existence of the first solution $U_0(P)$, nonpositive in case a) and small in the sense that

$$|U_0(P)| \leqslant \frac{1-\sqrt{1-4BK_1}}{2K_1},$$

in case b), was proved in papers (^{3},^{4}). Put now $U(P) = U_0(P) + W(P)$. Then the function $W(P)$ satisfies the following conditions:

$$\Delta W - 2U_0 W = W^2, \qquad W|_\Gamma = 0. \tag{4}$$

Let us use Theorem 1. Take as the function $q(x, y)$

$$q(x, y) = 2U_0(x, y).$$

In case a) the function $q(x, y)$ is nonpositive, and in case b) the function $q(x, y)$ is sufficiently small in absolute value, so that all eigenvalues of the kernel $K(P, Q)$ are positive and the first eigenfunction $\varphi_1(P) \geqslant 0$.

By virtue of the remark to Theorem 1, there exists a nontrivial function $W(P)$ satisfying conditions (4).

Let us pass to the study of the boundary-value problem (1) in the case of nonpositive values of the function $\varphi(s)$.

Lemma. There exists a constant C_* such that the boundary-value problem for the disk $r \leqslant R$

$$\Delta V = V^2, \qquad V(R) = C \tag{5}$$

has a solution if $C \geqslant C_*/R^2$, and has no solution if $C < C_*/R^2$.

This fact follows from the study of the family of solutions of the boundary-value problem (5) depending only on r. A numerical solution of the corresponding ordinary differential equation gives for the value C_* the approximate value $C_* = -1.40$.

From Theorem 2 it follows that in the closed domain $\overline{G}$ there exists a solution $V(P)$ of the equation $\Delta V = V^2$, taking on the boundary Γ negative values $\psi(s)$.

Theorem 4. If the function $\varphi(s)$ is such that

$$\psi(s) \leqslant \varphi(s) \leqslant 0,$$

then a solution of the boundary-value problem (1) exists.

Proof. Consider the following method of successive approximations:

$$\Delta U_k = U_{k-1}^2, \qquad U_k|_\Gamma = \varphi(s) \quad (k = 1, 2, \dots),$$

where $U_0(x, y)$ is the harmonic function corresponding to the boundary values $\varphi(s)$.

We shall show that the sequence of functions U_k is a monotone sequence. Indeed, for the functions $U_1(x, y)$ and $U_0(x, y)$ the inequality

$$U_1(x, y) \leqslant U_0(x, y),$$

holds, since

$$\Delta(U_1 - U_0) = U_0^2 \geqslant 0, \qquad U_1 - U_0|_\Gamma = 0.$$

Similarly, by induction we obtain the inequalities

$$U_k(x, y) \leq U_{k-1}(x, y) \qquad (k = 1, 2, \dots)$$

The sequence $\{U_k\}$ is bounded below. Indeed, for the function $U_1(x, y)$ we have

$$\Delta(V - U_1) = V^2 - U_0^2 \geq 0, \qquad V - U_1\big|_\Gamma = \psi(s) - \varphi(s) \leq 0;$$

therefore, everywhere in the domain G the inequality

$$V(x, y) \leq U_1(x, y)$$

holds. It is also not difficult to see that

$$V(x, y) \leq U_k(x, y) \qquad (k = 1, 2, \dots).$$

Writing the integral equation for the limiting function $U(x, y)$,

$$U(x, y) = \lim_{k\to\infty} U_k(x, y),$$

we obtain that the function $U(x, y)$ solves the boundary-value problem (1).

Corollary. The boundary-value problem (1) has a solution if

$$\frac{C_*}{R^2} \leq \varphi(s) \leq 0,$$

where R is the radius of a disk containing the domain G, and C_* is the quantity defined in the lemma.

Proof. By the lemma, for a disk of radius R there exists a solution of the boundary-value problem

$$\Delta V = V^2, \qquad V(R) = \frac{C_*}{R^2}.$$

This solution $V(P)$ assumes on the boundary Γ of the domain G values $\psi(s)$, which, by the subharmonicity of the function $V(P)$, satisfy the inequality

$$\psi(s) \leq \frac{C_*}{R^2}.$$

Theorem 5. *The boundary-value problem (1) has no solution if*

$$\varphi(s) < \frac{C_*}{R_0^2},$$

where R_0 is the radius of a disk contained in the domain G, and C_* is the quantity defined in the lemma.

Proof. Suppose the contrary. Then in the domain G there exists a solution of the boundary-value problem (1), which on the boundary Γ_0 of the disk of radius R_0 assumes values $\psi(s) \leq \varphi(s)$. Put $\max_{\Gamma_0} \psi(s) = A$ and consider the following boundary-value problem:

$$\Delta V = V^2, \qquad V(R_0) = A.$$

Since $\psi(s) \leq A \leq 0$, a solution of this boundary-value problem depending only on r exists by Theorem 4. But this contradicts the lemma, since $A < C_*/R_0^2$.

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Note: Figure translations are in progress. See original paper for figures.

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