THE INDEX IN BOUNDARY-VALUE PROBLEMS FOR INVARIANT ELLIPTIC SYSTEMS

1961

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Abstract

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MATHEMATICS

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THE INDEX IN BOUNDARY-VALUE PROBLEMS FOR INVARIANT ELLIPTIC SYSTEMS

(Presented by Academician S. L. Sobolev, 30 VI 1961)

In the general theory of solvability of boundary-value problems for linear elliptic equations and systems, three main questions may be singled out: the existence and uniqueness of a solution, normal solvability, and the index. Here the second question has recently been studied quite fully (see (^{1}) and the bibliography cited there), whereas the first and especially the third questions have been insufficiently studied. Thus, only very recently was an example constructed of a three-dimensional problem with nonzero index (^{2}).

Usually the index arises either as a certain characteristic of the boundary conditions for a fixed domain and equation (for example, the simplest case of the problem with oblique derivative), or as a characteristic of the matrix of coefficients of an equation (system) for a chosen domain and type of boundary conditions (^{3}). The aim of the present article is to show that, for a chosen special system of equations—the multidimensional analogue of the inhomogeneous Cauchy-Riemann system—and a fixed class of the simplest homogeneous boundary conditions (^{4}, ^{5}), the index is nothing other than the Euler characteristic of the chosen domain of Euclidean space (regarded as a manifold with boundary), computed for absolute homologies. This result is an immediate consequence of a special variant of Hodge' s theorem on harmonic forms, established in (^{6}). We note that, as it seems to the author, the considerations presented make quite transparent the global character of the question of the index and of the uniqueness conditions, showing that the absence of results in this direction when using methods that are essentially local (cf. (^{1})) is not accidental.

In what follows we shall use the basic definitions introduced in (^{4}, ^{5}). For convenience we reproduce the notation of the system of equations under consideration for the case of even n (n is the number of dimensions of the space):

$$\begin{array}{ll}
d\overset{0}{\omega} + \delta\overset{2}{\omega} = \overset{1}{\alpha}, & d\overset{1}{\omega} = \overset{0}{\alpha}, \\
d\overset{2}{\omega} + \delta\overset{4}{\omega} = \overset{3}{\alpha}, & d\overset{1}{\omega} + \delta\overset{3}{\omega} = \overset{2}{\alpha}, \\
\cdots\cdots\cdots \quad (\mathrm{K}) & \cdots\cdots\cdots \quad (\mathrm{K}^*) \\
d\overset{n-2}{\omega} + \delta\overset{n}{\omega} = \overset{n-1}{\alpha}. & d\overset{n-3}{\omega} + \delta\overset{n-1}{\omega} = \overset{n-2}{\alpha}, \\
 & d\overset{n-1}{\omega} = \overset{n}{\alpha}.
\end{array}$$

Let us recall that, upon passing (in a domain of Euclidean space) to the usual notation, we obtain a system of first-order equations with constant coefficients. And for odd n there remains that important feature of the equations under consideration that among the unknowns of system (K) there occur all covariants of even degree, and among the right-hand sides—all covariants of odd—

of even degree, and for the system (K*)—conversely. In what follows, in order not to encumber the exposition, we shall not explicitly specify the passages from a covariant to the corresponding differential form and back.

Let now V be a bounded domain in n-dimensional Euclidean space, which is a manifold with boundary S. It is assumed that S consists of a finite number of $(n-1)$-dimensional manifolds of class C^2. A homology basis for V in dimension p consists of absolute cycles $\overset{p}{\Gamma}_i$ $(i = 1, \dots, r_p)$ and relative cycles (or cycles $\mathrm{mod} S$), which we denote by $\overset{p}{C}_k$ $(k = 1, \dots, \beta_p)$. Taking a second copy V' of the domain V and identifying the corresponding boundary points, we obtain a manifold M without boundary. We shall regard M as a Riemannian manifold, assuming that the metric is induced by the Euclidean metric of the domain V. A homology basis in M may be chosen to consist of the cycles $\overset{p}{\Gamma}_i$ and the cycles $\overset{p}{C}_k = \overset{p}{C}{}^0_k + \overset{p}{C}{}^1_k$, where $\overset{p}{C}{}^1_k$ is the relative cycle of V', formed by the set of points corresponding to the points of $\overset{p}{C}{}^0_k$ (6).

Theorem (Hodge–Friedrichs). *For a given basis* $\overset{p}{\Gamma}_i, \overset{p}{C}_k$ *on the manifold* M *of the described structure, there exist harmonic forms* $\overset{p}{\varphi}_i, \overset{p}{\psi}_k$ *such that their periods satisfy the relations*

$$\begin{aligned}
&\int_{\overset{p}{\Gamma}_i} \overset{p}{\varphi}_j = \delta_i^j, \qquad \int_{\overset{p}{C}_k} \overset{p}{\varphi}_i = 0, \\
&\int_{\overset{p}{\Gamma}_i} \overset{p}{\psi}_k = 0, \qquad \int_{\overset{p}{C}_k} \overset{p}{\psi}_l = \delta_k^l,
\end{aligned} \tag{1}$$

where δ_k^l *is the Kronecker symbol. Here* $\overset{p}{\varphi}_i$ *belongs to the subspace of even* *covariants, while* $\overset{p}{\psi}_k$ *belongs to the subspace of odd covariants. A harmonic form all of whose periods are zero is identically zero.*

The proof of the stated theorem is outlined in (^{6}). Here, with respect to the metric tensor, only a Lipschitz continuity is assumed, as is the case in the situation under consideration.

We note that from the duality relations there follows the equality

$$r_p = \beta_{n-p}. \tag{2}$$

Every solution of the homogeneous system (K), (K*) is, obviously, a harmonic form. The converse is also true: every nonzero harmonic form $\overset{p}{\omega}$ gives a nontrivial solution of the homogeneous system (K) (p even) or (K*) (p odd). Indeed, it suffices to set all the remaining covariants entering the system identically equal to zero.

Theorem. *A necessary and sufficient condition for the solvability of the system* (K) *or* (K*) *in the subspace of even covariants is the fulfillment of the conditions*

$$\left(\overset{p}{\alpha}, \overset{p}{\varphi}_i\right) = 0, \qquad i = 1, \dots, r_p, \tag{3}$$

where p runs through all odd (all even) values. If, in addition, one requires that the conditions

$$\left(\overset{p}{\omega}, \overset{p}{\varphi}_i\right) = 0, \qquad i = 1, \dots, r_p, \tag{4}$$

be satisfied, where p runs through all even (all odd) values, then the solution will be determined uniquely.

Here by φ_i^p are meant the harmonic forms entering into the conditions of the Hodge–Friedrichs theorem, and solvability is understood in the sense of the existence of a generalized solution from W^1. To prove the theorem it is enough to observe that the collection of conditions (3), (4) for the covariant ω^p ensures the validity of the imbedding theorem:

$$|\omega^p, H|^2 \le C\big[(d\omega^p, d\omega^p) + (\delta\omega^p, \delta\omega^p)\big] \equiv C|\omega^p, W^1|^2,$$

after which the proof given in (^{4}) goes through without any changes.

In accordance with the usual definition of the index as the difference between the dimensions of the spaces of solutions of the given and the adjoint homogeneous problems, the index of the problem under consideration for the adjoint systems (K), (K^*) is equal to

$$I(V) = \sum_{p=0,\,2,\ldots} r_p - \sum_{p=1,\,3,\ldots} r_p = \sum_{p=0}^{n} (-1)^p r_p.$$

An analogous construction can obviously also be carried out in the subspace of odd covariants. But in this case, by virtue of (2), the index of the corresponding problem will always be the same.

Returning to the original domain V of Euclidean space, it remains only to observe that the boundary problems in V induced by considering the subspace of even (odd) covariants, i.e. with boundary conditions of the form (7), § 5 in ([4]) (or (8), ibid.), will be problems with index $I(V)$.

Remark 1. If, in studying the solvability of the systems (K), (K^*) on the manifold M, one does not restrict the consideration to the subspace of even (odd) covariants, then the index will be the ordinary Euler characteristic, and the cases of even and odd dimension n differ sharply: if the dimension is odd, then the index is always equal to zero. It is curious to note that a necessary condition for the possibility of specifying on M a metric tensor with Lorentz signature, under which the systems (K) become hyperbolic ([7]), is precisely the vanishing of the Euler characteristic.

Remark 2. In considering the simplest problems for the equation

$$-\Delta\omega^p \equiv (d\delta + \delta d)\omega^p = \alpha,$$

which has been the object of numerous investigations, a nonzero index, by virtue of the self-adjointness of Δ, never arises.

Example 1. The simplest problems in a domain homeomorphic to a ball, considered in ([4,5]), are, in a space of any dimension, problems with index 1, which once again emphasizes their similarity with the two-dimensional case, which is classical.

Example 2. Let the domain V have the form $1 \le r \le 2$, where $r^2 = (x^1)^2 + (x^2)^2 - (x^3)^2$. Then the forms given in V by the formulas

$$\varphi^2 = \frac{1}{4\pi} \frac{x^3\,dx^1 \wedge dx^2 - x^2\,dx^1 \wedge dx^3 + x^1\,dx^2 \wedge dx^3}{r^3},$$

$$\psi^1 = \frac{1}{2} \frac{x^1\,dx^1 + x^2\,dx^2 + x^3\,dx^3}{r},$$

will be continuous in $M = V + V'$, φ^2 under even, and ψ^1 under odd continuation, as is directly verified with the aid of the last group of formulas of § 7 in ([4]). Being harmonic, they also give, for the M under consideration, forms satisfying the equalities (1). The boundary problem with the conditions defined by the above-mentioned formulas from (4), has index 2.

Example 3. Let V be an n-dimensional ball with N balls of smaller radius removed, lying entirely inside V. Then for $V + V'$

$$r_0 = \beta_n = 1, \qquad \beta_1 = r_{n-1} = N,$$

while the remaining Betti numbers are zero, i.e. $I(V) = N + 1$. Thus, in a space of an arbitrary number of dimensions, for fixed systems of equations and a fixed class of boundary conditions, for any prescribed integer R one can indicate a domain in which $I(V) = R$.

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Received
29 VI 1961

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Note: Figure translations are in progress. See original paper for figures.

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