MATHEMATICS

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Abstract

Full Text

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ON SINGULAR GOURSAT PROBLEMS IN A NEIGHBORHOOD OF THE ZERO AND INFINITELY REMOTE SPECIAL CHARACTERISTIC

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Consider, in the domain D $(0 \leq x \leq x_0,\ 0 \leq y \leq y_0)$, the equation

$$L(z, B, C) = xz_{xy} + A(x)z_x + B(x)z_y + C(x)z = 0, \tag{1}$$

assuming that $A(x) > 0$, and that $B(x)$ and $C(x)$ are continuous together with their derivatives of the 2nd order on the interval X $(0 \leq x \leq x_0)$. Denote by $z(x, y, B, C)$ the solution of (1), twice continuously differentiable in D, for which

$$z(0, y) = f(y), \qquad z(x, 0) = 0, \qquad f(0) = 0. \tag{2}$$

Here, as in ([1]), we shall assume that $f(y)$ belongs to the class C_p^q on the segment Y $(0 \leq y \leq y_0)$ of the axis Oy.

In the study of the singular Goursat problem (1), (2), the following Duhamel principle plays an important role: if $U(x, y)$ is the integral of equation (1) with discontinuous boundary data $U(0, y) = 1$, $U(x, 0) = 0$, then

$$z(x, y, B, C) = D_y \int_0^y U(x, y - \eta) f(\eta)\, d\eta = \int_0^y U(x, y - \eta)\, df(\eta). \tag{3}$$

When $A(x) \equiv A(0) = a,\ B(x) \equiv B(0) = b,\ C(x) \equiv 0$, i.e. $z(x, y, B, C) = z(x, y, b)$ satisfies the equation (1)

$$L(z, b) = xz_{xy} + az_x + bz = 0 \qquad (a > 0), \tag{4}$$

we obtain $U(x, y)\Gamma(b) = \gamma(b, ay/x)$, where $\gamma(b, z)$ is Euler’ s incomplete gamma function. The existence of discontinuous solutions $U(x, y)$ in the general case (1) is proved by the method of successive approximations.

Fixing an arbitrary point $M(x_1, y_1) \in D$, construct the rectangle $G = OAMBO$ with vertices $O(0,0)$, $A(x_1, 0)$, $B(0, y_1)$. Let $z = 0$ on OA and OB. We shall show that then $z \equiv 0$ everywhere in $\overline{G}$. Indeed, multiply (1) by $2[xz_x + B(x)z]$, and then, integrating the result over the domain G, apply Green' s theorem. Then we obtain

$$\int_{AM}(AB+Cx)z^2\,dy+\int_{BM}(xz_x+Bz)^2\,dx+\iint_G\{2Axz_x^2+[2BC-(AB+Cx)_x]z^2\}\,dx\,dy=0.$$

Hence, under the conditions $A > 0,\ B \geq 0,\ C \geq 0,\ 2BC - (AB + Cx)_x \geq 0$, it follows that $z \equiv 0$ in $\overline{G}$. With the aid of connection formulas one can pass in (4) from negative b to positive b, and thereby extend the uniqueness theorem proved above also to parameters $b < 0$. As the estimates obtained show, for small values of y/x for equation (1), as in the case (4), $U = O[(y/x)^b]$; therefore the integrals in (3) converge only when $b > -1$.

Theorem 1. Let $V(x, y)$ satisfy the equation

$$xV_{xy} + AV_x + (B_2 - B_1)V_y + (C_2 - C_1)V = 0,$$

and the discontinuous initial conditions $V(x, 0) = 0,\ V(0, y) = 1$, and let $z_k = z(x, y, B_k, C_k)$ be the solutions of problem (2) for $L(z, B_k, C_k) = 0$ $(k = 1, 2)$. Then, if $b_2 - b_1 > -1$,

$$z_2(x,y) = D_y\int_0^y V(x,y-\eta)z_1(x,\eta)\,d\eta = \int_0^y V(x,y-\eta)z_{1\eta}(x,\eta)\,d\eta. \quad (5)$$

In this case the Riemann functions U_1, U_2, and V themselves are related by the equality

$$U_2(x,y-\eta) = D_y\int_\eta^y V(x,y-t)\,U_1(x,t-\eta)\,dt. \quad (6)$$

In the case (4), $V\Gamma(b_2-b_1) = \gamma(b_2-b_1, ay/x)$, and (6) reduces to the well-known addition theorem for the functions $\gamma(a, z)$. Since the conditions defining $V(x, y)$ are analogous to those under which the solutions $U(x, y)$ were constructed, the conclusions obtained for $U(x, y)$ carry over to $V(x, y)$, and, in particular, for small y/x one has here $V(x, y) = O[(y/x)^{b_2-b_1}]$. The equalities (5), by the same methods as the formulas (3), are generalized to the case when $b_2 - b_1 < -1$. For example, using the symbol of Riemann–Liouville fractional differentiation and assuming $f(y) \subset C_n^n(Y)$, we find, for $b = b_2 - b_1 \geq -n$ and any $n = 0, 1, 2, ...,$

$$z(x,y,b_2) = \left(x/a + D_y^{-1}\right)^{-b} D_y^{-(b+n)}\,[D_y^n z(x,y,b_1)]. \quad (7)$$

These series terminate when $b = -m$ $(m = 0, 1, 2, ...)$ and give an expansion corresponding to the recurrence relation $z(b) = \left(1 + \frac{x}{a} D_y\right)^m z(b+m)$ for contiguous functions $z(x, y, b + m)$ $(m = 0, 1, 2, ...)$ ([1]). To construct the transformation operators T_y^{-1}, inverse with respect to (5), it is enough to find from (5) $z_1(x, y)$, i.e., to solve (for $x = \text{const}$) Volterra integral equations of the first kind of convolution type with singular kernels $V(x, y - \eta)$. Since $z(x, y, 0) = f(y)$, the T_y^{-1} give, in particular, the inversion of the formulas (3) with respect to the initial function $f(y)$. Thus, for example,

$$f(y) = (x/a)^b \times \exp(-ay/x) D_y^{b-1} D_y \left[\exp(ay/x) z(x, y, b)\right].$$

Having obtained similar inversions $f(y) = T_y^{-1}[z(x, y, B, C)]$ for (1), one can then generalize (5) to the case when the initial values $f_1(y)$ and $f_2(y)$ of the integrals z_1 and z_2 are connected by a previously specified relation. For example,

$$z_2(x, y, b_1) = a^{b_1 - b_2} D_y D_y^{b_2 - b_1 - 1} z_1(x, y, b_1) \quad (b_2 > b_1 > 0), \tag{8}$$

if $f_1(y)$ and $f_2(y)$ satisfy the same equality. Along with T_y, it is important also to consider the transformation operators T_x, acting with respect to the singular variable x.

Theorem 2. Let $z(x, y, b + n + 1)$ be a solution of the problem $z(0, y) = (-a)^{-n-1} (b)_{n+1} f^{(n+1)}(y)$, $z(x, 0) = 0$ for the equation $L(z, b + n + 1) = 0$, where $f(y) \subset C_{n+3}^{n+1}$. Then the equality holds

$$z(b) = \sum_{k=0}^{n} (b)_k / k! \left(-\frac{x}{a}\right)^k f^{(k)}(y) + \frac{1}{n!} \int_0^x (x - \xi)^n z(\xi, y, b + n + 1)\, d\xi. \tag{9}$$

One further class of singular Goursat problems deserves attention, where zero initial values are prescribed on the infinitely distant regular characteristic $y = -\infty$. Namely, denote by $z_0(x, y, B, C)$ the integral of equation (1) for which

$$z_0(0, y) = f(y), \qquad \lim_{y \to -\infty} z_0(x, y) = 0, \qquad f(-\infty) = 0. \tag{10}$$

Assuming that $f(y)$ is given everywhere on the half-line $-\infty < y < y_0$, we obtain, for $b > 0$,

$$z_0(x, y, b) = \frac{1}{\Gamma(b)} \left(\frac{a}{x}\right)^b \int_{-\infty}^{y} (y - \eta)^{b-1} \exp\left[-\frac{a(y - \eta)}{x}\right] f(\eta)\, d\eta, \tag{11}$$

and when $f(y) \in C_n^n(-\infty < y < y_0)$, $b > -n - 1$,

$$z_0 = \sum_{k=0}^{n} (b)_k/k! \left(-\frac{x}{a}\right)^k f^{(k)}(y) + R_n,$$

where

$$a^n \Gamma(b) R_n = \int_0^\infty \exp(-a\xi)\Psi(1-b, 1-b-n; a\xi)\, f^{(n+1)}(y - x\xi)\, d\xi.$$

For z_0 there hold relations analogous to those satisfied by $z(x, y, b)$. From (3) it follows that

$$\lim_{x\to\infty} [x^{bz}(x, y, b)] = a^b D_y^{-b} f(y) = \Phi(y).$$

Thus, $z(x, y, b)$ is an integral of equation (4), regular in a neighborhood of the singular characteristic $x = \infty$, with the exponent of this singular line equal to b. Therefore, in order to study $z(x, y, b)$ in a neighborhood of $x = \infty$, it is convenient to introduce the function $u(x, y, b) = x^{-b} z(1/x, y, b)$ and thereby reduce the considerations to the Goursat problem:

$$\mathcal{L}(u, b) = u_{xy} + axu_x + abu = 0, \tag{12}$$

$$u(0, y) = \Phi(y), \qquad u(x, 0) = 0, \qquad \Phi(0) = 0. \tag{13}$$

Its solution is obtained from (3), if one passes from $f(y)$ to $\Phi(y)$, and has the form:

$$u(x, y, b) = D_y \int_0^y U(x, y-\eta)\Phi(\eta)\, d\eta = \int_0^y U(x, y-\eta)\, d\Phi(\eta), \tag{14}$$

where this time $U(x, y) = {}_1F_1(b, 1, -axy)$. Conversely, (3) arises from (14) and the relation $\Phi(y) = a^b D_y^{-b} f(y)$. For $b = 1$, $b = 1/2$, $b = -m$, and $b = 1 + m$ ($m = 0, 1, 2, \ldots$), ${}_1F_1$ in formulas (14) is replaced by the resolvents $\exp(-ar)$, $\exp(-ar/2) I_0(ar/2)$, $L_m(-ar)$, and $\exp(-ar) L_m(ar)$ ($r = x(y-\eta)$), respectively. Setting in (12) $ab = k^2$, and then passing to the limit as $a \to 0$, we arrive, for $w = \lim_{a\to 0} u(x, y, k^2/a)$, at the telegraph equation

$$w_{xy} + k^2 w = 0.$$

The same limiting transition in (14) gives

$$U(x, y) = \lim_{a\to 0} {}_1F_1(k^2/a, 1, -axy) = J_0(2k\sqrt{xy}).$$

In the form (14) one also writes the solution of problem (13) for the more general equation

$$\mathcal{L}(u,b) = u_{xy} + A(x)u_x + B(x)u = 0, \tag{15}$$

where $A(x)$ and $B(x) \in C_2(X)$, and $A(x) = O(x)$ as $x \to 0$. Using (12) as a majorant, one can show that here $U(x,y)$ is represented by the uniformly and absolutely convergent series

$$U(x,y) = \sum_{n=0}^{\infty} U_n(x) y^n,$$

in which $U_n(x)$ are expressed in terms of $A(x)$ and $B(x)$. Moreover, if $|\Phi(y)| \le K$ on Y, and on the segment X, $A(x) = xa(x)$, $|a(x)| \le M$, $|B(x)| \le MN$, where K, M, and N are positive constants, then

$$|u| \le K\,{}_1F_1(N,1,Mxy), \qquad (x,y) \in D.$$

The same estimate, but with $K = 1$, also holds for $U(x,y)$. Considering solutions $u_k = u(x,y,B_k)$ of the equations $\mathcal{L}(u,B_k) = 0$ $(k = 1,2)$, we again arrive at equalities (5), but now the kernel $V(x,y)$ is determined from

$$V_{xy} + AV_x + (B_2 - B_1)V = 0$$

and the initial conditions

$$V(x,0) = V(0,y) = V(0,0) = 1,$$

which contain no jumps; hence $V(x,y)$ is continuous in $\overline{D}$ and is computed by means of the same type of series as $U(x,y)$. In the case (12),

$$V = {}_1F_1(b_2 - b_1, 1, -axy),$$

and to the solution w there corresponds

$$V = J_0\left[2\sqrt{(k_2^2 - k_1^2)xy}\right].$$

For (15), if $|B_2 - B_1| \le MN_0$, then

$$|V| \le {}_1F_1(N_0, 1; Mxy)$$

in D; moreover, here too U_1, U_2, V are related by formula (6), which in the particular cases indicated above ⋯

reduce in special cases to the known integrals of Bateman–Erdelyi and Sonin with the functions ${}_1F_1$, $J_0(z)$ ([2]).

By means of transformation (8) one can also pass directly from formulas (5) for $z(x, y, b_k)$ to the corresponding relations for $u(x, y, b_k)$, and conversely. More general transformation operators of convolution type, where $\Phi_2(y)$ may differ from $\Phi_1(y)$, can be constructed if, putting $x = \text{const}$, one inverts the Volterra integral equations (14). Thus, for example,

$$\Phi(y) = \exp(-axy) \int_0^y {}_1F_1(b, 1; ar) D_\eta[\exp(ax\eta) u(x, \eta, b)]\, d\eta,$$

$$\Phi(y) = \int_0^y I_0(2k\sqrt{r})\, w_\eta(x, \eta, k)\, d\eta.$$

Similar relations for the example are constructed when $\Phi_2(y) = P(y)\Phi_1(y)$, where $P(y)$ is an arbitrary weight function; moreover, together with the integral forms one obtains the expansions

$$u_2 = \nabla_a^{-b_2} [P(y) \exp(-axy) \nabla_{-a}^{-b_1} (e^{axy} u_1)],$$

$$w_2 = \chi_2 [P(y) \chi_1^{-1} w_1],$$

$$u_2 = \nabla_a^{-b_2} [P(y) \chi_1^{-1} w_1],$$

$$w_2 = \chi_2 [P(y) \exp(-axy) \nabla_{-a}^{b_1} (e^{axy} u_1)],$$

where $\nabla_a = 1 + axD_y^{-1}$, $\chi = \exp(-k^2 x D_y^{-1})$. The functions $u(x, y, b_i)$, $w(x, y, k_i)$ are also related with respect to the variable x. For example,

$$u(b+n)(b)_n = x^{1-b} D_x^n [x^{b+n-1} u(b)] \quad (n = 1, 2, ...),$$

and, for

$$b > 0, \quad k \neq 0 \quad u(x, y, b) = \frac{1}{\Gamma(b)} \left(\frac{k^2}{a}\right)^b \int_0^\infty \xi^{b-1} \exp\left(-\frac{k^2 \xi}{a}\right) w(x\xi, y, k)\, d\xi.$$

In addition, for $u(x, y, b_k)$ the relations (9) and equalities of the form $u_2 = T_x[u_1]$, found in ([1]) (see in ([1]), (6) and (11)), are fulfilled.

We arrive at similar conclusions also by studying the solutions $u_0(x, y, b_k)$ of problem (10), (12). Let us further note that (12) possesses the following property: if $u_1(x, y, a, b)$ is an integral of this equation, then the function $u_2 = \exp(-axy)u(y, x, -a, 1-b)$ also satisfies it. By means of this transformation, the results obtained for $u(x, y, b)$ are transferred to the solution $\tilde{u}(x, y, b)$ of the Goursat problem $\tilde{u}(x, 0) = \Phi(x)$, $\tilde{u}(0, y) = 0$, $\Phi(0) = 0$. Finally, denote by $v(x, y, b)$ a solution of the parabolic-type equation

$$xv_{xx} + bv_x - av_y = 0 \quad (a > 0)$$

with boundary data (2) ([1,3,4]). The function $v(x, y, b)$ is also represented in the form of a Duhamel integral (3) with kernel $\Gamma(1-b)U(x, y) = \Gamma(1-b, ax/y)$.

Theorem 3. If

$$c_1\Gamma(b)\Gamma(1-b_2) = \Gamma(1-b_1),$$

$$c_2\Gamma(1-b_1)\Gamma(1+\bar{b}) = -\Gamma(1-b_2)$$

and for arbitrary noninteger values $b_1 > 0$, $b_2 > 0$, $b = b_2 - b_1 > 0$, $\bar{b} = b_1 - b_2 > -1$, the relations

$$v(x, y, b_2) = c_1 \int_1^\infty \xi^{b_1-1}(\xi-1)^{b-1} v(x\xi, y, b_1)\, d\xi;$$

$$v(x, y, b_1) = c_2 x^{1-b_1} D_x \int_x^\infty \xi^{b_2-1}(\xi-x)^{\bar{b}} v(\xi, y, b_2)\, d\xi,$$

hold, and for $m = 1, 2, \ldots$

$$v(b)\Gamma(m+b) = \Gamma(b)x^{1-b} D_x^m [x^{m+b-1} v(m+b)].$$

These formulas define transformation operators of Delsarte type $\mathfrak{B}_x$ and $\mathfrak{B}_x^{-1}$, satisfying the identities

$$\mathfrak{B}_x L_x^{(1)} f(x) = L_x^{(2)} \mathfrak{B}_x f(x),$$

$$\mathfrak{B}_x^{-1} L_x^{(2)} f(x) = L_x^{(1)} \mathfrak{B}_x^{-1} f(x),$$

$$L_x^{(1)} f(x) = \mathfrak{B}_x^{-1} L_x^{(2)} \mathfrak{B}_x f(x),$$

$$L_x^{(2)} f(x) = \mathfrak{B}_x L_x^{(1)} \mathfrak{B}_x^{-1} f(x),$$

if $f(0) = f(\infty) = 0$, $L_x^{(k)} f = x f_{xx} + b_k f_x$ $(k = 1, 2)$.

Analogous results are obtained also for the integral $v_0(x, y, b)$ with boundary data (10).

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Note: Figure translations are in progress. See original paper for figures.