ON THE THEORY OF FREE FINITE OSCILLATIONS OF THE INTERFACE BETWEEN TWO UNBOUNDED HEAVY FLUIDS OF DIFFERENT DENSITIES

1961

SovietRxiv

Abstract

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HYDROMECHANICS

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ON THE THEORY OF FREE FINITE OSCILLATIONS OF THE INTERFACE BETWEEN TWO UNBOUNDED HEAVY FLUIDS OF DIFFERENT DENSITIES

(Presented by Academician V. V. Shuleikin, 16 VII 1960)

Let us consider the plane motion of an unbounded ideal incompressible heavy fluid consisting of two layers of different density, arranged one above the other. Denote the density of the upper layer by ρ_1, and that of the lower by ρ_2 ($\rho_1 < \rho_2$). In what follows, for all quantities referring to the upper layer we shall put the first subscript 1, and for the lower layer the subscript 2. We have studied two-dimensional free finite oscillations of the surface (line) of separation between the two layers of the fluid under consideration, and here we briefly set forth the results obtained by us. In doing so, we generalize the method of solution by means of Lagrangian variables, applied by us, for example, in papers ([1−3]) to the case of a standing wave on the surface of one heavy fluid of both finite and infinite depth. In the literature known to us, we have not encountered works with a solution of the problem considered.

In the plane of motion we choose a rectangular coordinate system xOy; we make the axis Ox coincide with the straight horizontal line of separation of the two layers in the fluid at rest; we direct the axis Oy vertically upward.

We solve the problem in Lagrangian variables a_i and b ($i = 1, 2$). We seek $x_i(a_i, b, t)$, $y_i(a_i, b, t)$–the coordinates of an individual moving particle in both layers of the fluid–and functions $Q_i(a_i, b, t)$ connected with the pressure $p_i(a_i, b, t)$ by the formula $Q_i = -p_i/\rho_i - gy_i$, where g is the acceleration of gravity. We require that at $t = 0$, i.e. at the initial instant, and consequently throughout the entire time of motion, the surface (line) of separation correspond to the value $b = 0$. We assume that a_i and b are, in each layer, curvilinear coordinates of the fluid particles at the initial instant. In this case let the lines $a_i = \text{const}$ be straight lines parallel to the axis Oy; let the lines $b = \text{const}$ be curves forming the second family of lines of the coordinate net, having an initially unknown form.

Assuming that at $t = 0$ the equations of the second family of coordinate curves in both layers have the form

$$y_i = b + f_i(x_i, b), \qquad (1)$$

where $f_i(x_i, b)$ are, by condition, unknown functions sought, we must, evidently, require the fulfillment of the following initial conditions at $t = 0$:

$$x_i(a_i, b, 0) = a_i; \qquad y_i(a_i, b, 0) = b + f_i(a_i, b). \qquad (2)$$

On the surface (line) of separation, for $b = 0$, the following boundary conditions must be satisfied.

First, equality of pressures must hold:

$$p_1(a_1, 0, t) = p_2(a_2, 0, t). \qquad (3)$$

Secondly (the kinematic condition), the particles of both layers of the fluid cannot, during the motion, leave the surface (line) of separation, or, in other words, the points of the interface surface belong throughout the motion to both layers of the fluid, i.e., we require that

$$x_1(a_1, 0, t) = x_2(a_2, 0, t), \qquad (4)$$

$$y_1(a_1, 0, t) = y_2(a_2, 0, t). \qquad (5)$$

In satisfying the boundary conditions it turned out that the variables a_1 and a_2 on the interface surface, i.e. for $b = 0$, must be regarded as being related to each other, and that this relation changes with time.

We seek standing oscillations, though of finite but small amplitude and differing little from the oscillations of the linear theory. Therefore we introduce a small parameter ε and make the substitution, putting

$$x_i = a_i + \varepsilon\xi_i, \qquad y_i = b + \varepsilon\gamma_i + \varepsilon\eta_i;$$

$$f_i = \varepsilon\gamma_i, \qquad Q_i = \varepsilon q_i \quad (i = 1, 2);$$

here $\xi_i(a_i, b, t, \varepsilon)$, $\eta_i(a_i, b, t, \varepsilon)$, $\gamma_i(a_i, b, \varepsilon)$, $q_i(a_i, b, t, \varepsilon)$ are new unknown functions. For these functions, from the equations of hydrodynamics we obtain the differential equations

$$\frac{\partial^2 \xi_i}{\partial t^2} + \varepsilon \left[\frac{\partial^2 \xi_i}{\partial t^2} \frac{\partial \xi_i}{\partial a_i} + \frac{\partial^2 \eta_i}{\partial t^2} \frac{\partial(\eta_i + \gamma_i)}{\partial a_i} \right] = \frac{\partial q_i}{\partial a_i},$$

$$\frac{\partial^2\eta_i}{\partial t^2}+\varepsilon\left[\frac{\partial^2\xi_i}{\partial t^2}\frac{\partial\xi_i}{\partial b}+\frac{\partial^2\eta_i}{\partial t^2}\frac{\partial(\eta_i+\gamma_i)}{\partial b}\right]=\frac{\partial q_i}{\partial b}, \tag{7}$$

$$\frac{\partial\xi_i}{\partial a_i}+\frac{\partial\eta_i}{\partial b}=-\varepsilon\left[\frac{\partial\xi_i}{\partial a_i}\frac{\partial(\eta_i+\gamma_i)}{\partial b}-\frac{\partial\xi_i}{\partial b}\frac{\partial(\eta_i+\gamma_i)}{\partial a_i}\right].$$

The initial conditions (2) will take the form:

$$\xi_i(a_i,b,0,\varepsilon)=0, \qquad \eta_i(a_i,b,0,\varepsilon)=0. \tag{8}$$

To satisfy the boundary conditions, we suppose that on the interface surface

$$a_1=a+\varepsilon\mu(a,t,\varepsilon), \qquad a_2=a+\varepsilon\nu(a,t,\varepsilon); \tag{9}$$

here a is an auxiliary parameter; the functions μ and ν are unknown and are determined from the boundary conditions and from the requirement

$$\mu(a,0,\varepsilon)=\nu(a,0,\varepsilon)=0. \tag{10}$$

Taking into account that on the surface (line) of separation $x_1=x_2=a$, we write the boundary conditions (4), (5), and (3) in the form:

$$a_1+\varepsilon\xi_1(a_1,0,t,\varepsilon)=a, \qquad a_2+\varepsilon\xi_2(a_2,0,t,\varepsilon)=a; \tag{11}$$

$$\eta_1(a_1,0,t,\varepsilon)+\gamma_1(a_1,0,\varepsilon)=\eta_2(a_2,0,t,\varepsilon)+\gamma_2(a_2,0,\varepsilon); \tag{12}$$

$$\rho_1\{q_1(a_1,0,t,\varepsilon)+g[\gamma_1(a_1,0,\varepsilon)+\eta_1(a_1,0,t,\varepsilon)]\}$$

$$=\rho_2\{q_2(a_2,0,t,\varepsilon)+g[\gamma_2(a_2,0,\varepsilon)+\eta_2(a_2,0,t,\varepsilon)]\}. \tag{13}$$

The solution of the linear problem has the form

$$q_{10}=D_{10}\sin K_0a_1e^{-K_0b}\sin\sigma t, \qquad \xi_{10}=-\frac{K_0D_{10}}{\sigma^2}\cos K_0a_1e^{-K_0b}\sin\sigma t,$$

$$\eta_{10}=\frac{K_0D_{10}}{\sigma^2}\sin K_0a_1e^{-K_0b}\sin\sigma t, \qquad \gamma_{10}\equiv 0, \qquad \mu_0=\frac{K_0D_{10}}{\sigma^2}\cos K_0a\sin\sigma t,$$

$$q_{20} = D_{20} \sin K_0 a_2 e^{K_0 b} \sin \sigma t, \qquad \xi_{20} = -\frac{K_0 D_{20}}{\sigma^2} \cos K_0 a_2 e^{K_0 b} \sin \sigma t, \quad (14)$$

$$\eta_{20} = -\frac{K_0 D_{20}}{\sigma^2} \sin K_0 a_2 e^{K_0 b} \sin \sigma t, \qquad \gamma_{20} \equiv 0, \qquad \nu_0 = \frac{K_0 D_{20}}{\sigma^2} \cos K_0 a \sin \sigma t;$$

$$D_{20} = -D_{10}, \qquad K_0 = \frac{\sigma^2}{g} \frac{\rho_2 + \rho_1}{\rho_2 - \rho_1}. \quad (15)$$

here σ is the prescribed frequency of oscillations; $D_{10} = \text{const}$ is a fixed constant.

Let $K = 2\pi/L$; L is the period with respect to the variables a_i. In solving the nonlinear problem we regard $K = K(\sigma, \varepsilon)$ as an unknown function, determined in the course of solving the problem and satisfying the condition

$$\lim_{\varepsilon=0} (K - K_0) = 0. \quad (16)$$

We show that, as a result, we arrive at the following mathematical problem:

Determine the functions $q_i, \xi_i, \eta_i, \gamma_i, \mu, \nu$ $(i = 1, 2)$ and the constant $K = K(\sigma, \varepsilon)$ so that:

1) the differential equations (7), the initial conditions (8), (10), and the boundary conditions (11), (12), (13) are satisfied;
2) the required functions are periodic in a_1, a_2 and a with period $L = 2\pi/K$, periodic in t with period $T = 2\pi/\sigma$, where σ is a prescribed arbitrary quantity; bounded as $b \to +\infty$ for the upper fluid and as $b \to -\infty$ for the lower fluid;
3) the required functions and the constant, for $\varepsilon = 0$, take the values corresponding to the linear problem (they have one subscript zero) and are determined by formulas (14), (15), and (16);
4) the functions q_1 and q_2 satisfy normalization conditions, i.e. the differences $q_1 - q_{10}$ and $q_2 - q_{20}$ contain the expressions $M_1 \sin K_0 a_1 e^{-K_0 b} \sin \sigma t$ and $M_2 \sin K_0 a_2 e^{K_0 b} \sin \sigma t$ only under the condition $\rho_1 M_1 = M_2 \rho_2$; the functions $\gamma_i(a_i, b, \varepsilon)$ are harmonic in a_i and b; the functions q_1 do not contain terms $D' + D'' f(t)$, if D' and D'' are constants (these conditions are somewhat formal, but they ensure sufficient symmetry and uniqueness of the solution);
5) the motion of the fluid in each of the layers has a velocity potential;

6) the functions ξ_1 and ξ_2 vanish for $a_1 = a_2 = \pi/2K$ and for any values of b, t, ε; both arguments a_1 and a_2 on the interface line ($b = 0$) take the common value $\pi/2K$ for $a = \pi/2K$ and any t and ε (by periodicity these conditions are also satisfied for $a = a_i = 5\pi/2K$).

Condition 6) ensures the absence of transport of liquid masses along the Ox axis, i.e. the oscillations being studied will be of the standing-wave type.

In solving the problem we seek the functions and the constant in the form of power series in ε. We show the possibility of determining any approximation. In constructing the solution, in order to avoid secular terms, we make the change of variables: $a' = Ka_i,\ a' = Ka$, and $b' = Kb$.

We present the results of computing the first three approximations. We have

$$K = \frac{\sigma^2}{g}\frac{(\rho_2+\rho_1)}{(\rho_2-\rho_1)} + \zeta^2\frac{\sigma^6}{4g^3}\frac{(\rho_2+\rho_1)(\rho_2^2+\rho_1^2)}{(\rho_2-\rho_1)^3} \qquad \left(\zeta = \varepsilon\frac{D_{10}}{g}\frac{(\rho_2+\rho_1)}{(\rho_2-\rho_1)}\right) \tag{17}$$

Putting $b = 0$ in the expressions for x_i and y_i, and eliminating a_i, we obtain the approximate equation of the surface (line) of separation

$$y_i(x_i, \varepsilon, t) = \zeta \sin x_i \sin\sigma t - \frac{\zeta^2(\rho_2-\rho_1)}{2(\rho_2+\rho_1)}\cos 2x_i \sin^2\sigma t +$$

$$+\frac{\zeta^3}{32}\frac{1}{(\rho_2+\rho_1)^2}\{[(7\rho_1^2+7\rho_2^2-10\rho_1\rho_2)\sin x_i +$$

$$+3(10\rho_1\rho_2 - 3\rho_1^2 - 3\rho_2^2)\sin 3x_i]\sin\sigma t +$$

$$+[2(\rho_1^2+\rho_2^2-\rho_1\rho_2)\sin x_i + (3\rho_1^2+3\rho_2^2-10\rho_1\rho_2)\sin 3x_i]\sin 3\sigma t\}; \tag{18}$$

in this case $K = 1$, and σ must be determined from (17).

In conclusion we indicate the features of the nonlinear oscillations studied here, which follow from the analysis of the approximate formulas and from the complete series giving the solution.

1. There are no fixed nodes; the nodes of the linear wave $x_i = 0$ and $x_i = \pi$ ($i = 1, 2$) move, in the first approximation, respectively according to the laws $x_i = {}^1/_2\zeta_1 \sin\sigma t$, $x_i = \pi - {}^1/_2\zeta_1 \sin\sigma t$, where $(\rho_2-\rho_1)\zeta_1 = \zeta(\rho_2+\rho_1)$. This motion takes place along a certain initial curve (see property 4)).
2. The extreme amplitudes (antinodes) will be at the points $x_i = \pi/2$ (crest), $x_i = 3\pi/2$ (trough) ($i = 1, 2$) and at $t = \pi/2\sigma$. At these points there are the same fluid particles, which execute only vertical oscillations.

3. The ordinates of the crests of the interface are greater in absolute value than the ordinates of the troughs—the curve of the line resembles a trochoid.

4. At $t = 0$ the equation of the interface approximately has the form

$$y_i = \zeta^4 \left(\widetilde{E}_{241} \cos 2x_i + \widetilde{E}_{242} \cos 4x_i \right).$$

Since $\widetilde{E}_{241} \neq 0$ and $\widetilde{E}_{242} \neq 0$, and in view of the arbitrariness of the initial instant of time, the interface in standing oscillations is never straightened.

5. The fluid particles situated on the interface move along this line; moreover, the particles of the upper and lower fluids move in opposite directions. This follows from the expressions for a_i in terms of a and t on the interface.

The fulfillment, in any approximation, of points 5) and 6) of the basic mathematical problem has been proved.

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Received
14 VII 1960

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.