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Abstract

Full Text

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A Criterion for the Reflexivity of a Banach Space

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The terminology and notation of this article coincide with the terminology adopted in ([1]).

Let M' denote a vector subspace of the space E', conjugate to some Banach space E, which is everywhere dense in the weak topology $\sigma(E', E)$. As is known ([1,2]), the **characteristic** of the subspace M' is the least upper bound of the numbers t such that the weak closure of the intersection of M' with the unit ball $S'_1 \subset E'$ contains the ball S'_t of radius t.

Suppose that on a Banach space E there is given a certain separable locally convex topology T, which is majorized by the original topology of the space E. Then the space E'_T, conjugate to the space E endowed with the topology T, will obviously be a vector subspace of E'.

The subspace E'_T is total, i.e., everywhere dense in the weak topology $\sigma(E', E)$. Indeed, since the topology T is separable and locally convex, for every element $x_0 \in E$ there exists a seminorm $p(x)$ from the set of seminorms defining the topology T such that $p(x_0) \neq 0$. From the Hahn–Banach theorem on the extension of a linear functional in a separable locally convex space it follows that there exists a linear functional x'_T, continuous in the topology T, for which $\langle x_0, x'_T \rangle = p(x_0) \neq 0$. Hence it follows that E'_T is total in E'.

Lemma 1. *In order that the unit ball S_1 ($\|x\| \leq 1$) of the space E be closed in the topology T, it is necessary and sufficient that the characteristic of the subspace $E'_T \subset E'$ be equal to one.*

Proof. If r denotes the characteristic of the subspace $E'_T \subset E'$, then $1/r$ is equal to the least upper bound of the numbers $\|x\|$, where x runs through the closure of the ball S_1 in E in the topology $\sigma(E, E'_T)$ ([1,2]). The proof of Lemma 1 is now obtained from the statement that the closure of a convex set is the same in all separable locally convex topologies on E consistent with the duality between E and E'_T.

Theorem 1. *In order that a Banach space E be reflexive, it is necessary and sufficient that its unit ball be a closed set in every separable locally convex topology T given on E and comparable with the original topology of the Banach space E.*

The proof of the necessity of this assertion is contained in ([3]) (see the proof of Lemma 3.3). Moreover, it is obtained at once from Lemma 1 if one takes into account the fact that E'_T, being a dense subspace of E' in the topology $\sigma(E', E)$, will be dense in E' also in the topology $\sigma(E', E'')$, since for a reflexive space these two topologies coincide. But a vector subspace of E' dense in the topology $\sigma(E', E'')$ will be a dense subspace also in the strong topology of the space E', which means that the characteristic of E'_T is equal to one.

To prove sufficiency we shall use the following formula for computing the characteristic of the vector subspace $E'_T \subset E'$:

$$r = \inf \frac{\|x + z''\|}{\|x\|}, \tag{1}$$

where z'' ranges over $(E'_T)^0$, and x over the set of all nonzero points of E ([1,2]).

Suppose that E is nonreflexive. It is not hard to see that in this case there exist elements $x \in E$ ($\|x\| = 1$) and $z'' \notin E$ in E'' such that x is not orthogonal to the subspace $\{\lambda z''\}$. This means that

$$1 > \inf_{\substack{z''_1 = \lambda z'' \\ x \in E,\ \|x\|=1}} \|z''_1 - x\| = \inf_{\substack{z''_1 - \lambda z'' \\ x \in E}} \frac{\|z''_1 - x\|}{\|x\|} = r. \tag{2}$$

Denote by M' the polar of z'' in the space E'. From (1) and inequality (2) it follows that M' has characteristic less than one. But $z'' \notin E$; therefore M' is a vector subspace of E', everywhere dense in the weak topology $\sigma(E', E)$, and hence it follows that M' and E are put into duality by the bilinear form $B(x, x') = \langle x, x' \rangle$, where $x \in E$, $x' \in E'$. Therefore $\sigma(E, M')$ is a separated locally convex topology which, obviously, is majorized by the original topology of the Banach space E. The unit ball S_1 of the space E is not a closed set in this topology, since its closure contains elements with norm arbitrarily close to $1/r > 1$. The theorem is proved.

The Mackey topology $\tau(E, M')$ is not normable. However, the following is true:

Lemma 2. *If E is a separable Banach space and M' is a strongly closed vector subspace of E', everywhere dense in the weak topology $\sigma(E', E)$, then the Mackey topology $\tau(E, M')$ majorizes some normed topology defined on the space E.*

Proof. In the space E', conjugate to a separable Banach space and endowed with the weak topology $\sigma(E', E)$, the unit ball S'_1 is a separable metrizable space.

The set $S'_1 \cap M'$, being a metrizable space in the topology $\sigma(M', E)$, is separable in this topology. Let $e'_1, \dots, e'_n, \dots$ be a total countable set of elements from $S'_1 \cap M'$ in the topology $\sigma(M', E)$. It is not hard to construct a balanced convex set $W \subset M'$ which is compact in the strong topology of the space E' and absorbs each of the one-point sets e'_n. W will be compact in the weak topology

$\sigma(E', E)$; therefore the polar W^0 in the space E will belong to the fundamental system of neighborhoods of zero for the Mackey topology $\tau(E, M')$.

The set W is total in M' and, consequently, in E' in the topology $\sigma(E', E)$. Hence it follows that W^0 contains no linear manifold. Taking now W^0 as the unit ball of a normed topology defined on E, we obtain the required normed topology.

From Theorem 1 and Lemma 2 it follows:

Theorem 2. *For a Banach space E to be reflexive, it is necessary and sufficient that its unit ball be closed in every normable topology defined on E and comparable with the original topology of the Banach space E.*

Proof. The assertion of the theorem for a separable Banach space E follows immediately from Theorem 1 and Lemma 2.

Let now E be a nonseparable nonreflexive Banach space. Then one can find in it a separable nonreflexive subspace E_1. In this subspace, by virtue of the preceding, one can introduce a new norm $\|x\|_1$, smaller than the original one, such that the ball $S_1 \cap E_1$ is not closed in this norm.

Denote by U_1 the set of all elements of E_1 such that $\|x\|_1 \leqslant 1$, and by U the convex hull $U_1 \cup S_1$. Then the gauge function $p(x)$ constructed from the set U gives rise in the space E to a norm smaller than the original one and coinciding with the norm $\|x\|_1$ on the subspace E_1. In the norm $p(x)$ the ball S_1 is not closed. The theorem is proved.

Using the results of (4), Theorem 2 can be formulated in the following form:

Theorem 3. *In order that a Banach space E be related to every Banach space E_0 into which it is normally embedded, it is necessary and sufficient that the space E be reflexive.*

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Note: Figure translations are in progress. See original paper for figures.