MATHEMATICS

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Abstract

Full Text

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ON SOME CLASSES OF SETS AND THEIR APPLICATIONS

1°. Let W denote some property defined with respect to all closed bounded sets in the plane, and let $\mathfrak{M}_W$ be the totality of all closed bounded sets possessing the property W.

The property W, and the class $\mathfrak{M}_W$ determined by this property, will be called **regular** if the following conditions are fulfilled:

1. **Monotonicity.** Every closed subset of a set from $\mathfrak{M}_W$ belongs to $\mathfrak{M}_W$.
2. **Additivity.** The sum of a finite number of pairwise disjoint sets from $\mathfrak{M}_W$ belongs to $\mathfrak{M}_W$.

Consider an arbitrary closed set E. A point $\xi \in E$ will be called a W-point of E if the intersection of E with any closed disk centered at ξ does not possess the property W, i.e. does not belong to $\mathfrak{M}_W$. The totality of all W-points is a closed subset of E, which we shall denote by $E_W^{(1)}$ and call the W-derived set of E of first order. If α is a positive integer or a transfinite number of the first kind and the set $E_W^{(\alpha-1)}$ is defined, then by the W-derived set of E of order α, $E_W^{(\alpha)}$, we shall mean the totality of all W-points of the set $E_W^{(\alpha-1)}$. If α is a transfinite number of the second kind and all $E_W^{(\beta)}$, $\beta < \alpha$, are defined, then as the W-derived set of E of order α we take the intersection of all $E_W^{(\beta)}$, $\beta < \alpha$.

By Baire' s theorem, the monotonically decreasing sequence of closed sets

$$E \equiv E_W^{(0)},\ E_W^{(1)}, \dots,\ E_W^{(n)}, \dots,\ E_W^{(\omega)}, \dots,\ E_W^{(\alpha)}, \dots$$

is stationary: beginning with some $\gamma < \Omega$, all $E_W^{(\beta)}$, $\beta \geqslant \gamma$, coincide with a certain set E_W, which we shall call the W-kernel of E. This set is identical with its W-derived set of first order, i.e. it is a W-perfect set, and every W-perfect subset of E belongs to E_W.

If E_W is the empty set, then E will be called W-reducible. Any set from $\mathfrak{M}_W$ is, obviously, W-reducible. The proposition formulated below in a certain sense reverses this circumstance and is valid for an arbitrary closed E and any regular class $\mathfrak{M}_W$.

Lemma. The set E is representable in the form

$$E = E_W + \sum_{n=1}^{\infty} E_n, \qquad E_n \subset E_{n+1},\ E_n \in \mathfrak{M}_W.$$

Corollary 1. In order that E be W-reducible, it is necessary

and it is sufficient that E be the sum of a finite or countable number of sets from $\mathfrak{M}_W$.

Corollary 2. Any one of the following three conditions is equivalent to the W-reducibility of E:

1. Among the closed subsets of E there are no W-perfect sets.
2. The first W-derivative of any closed $F \subset E$ is nowhere dense in F.
3. Every closed subset of E contains a portion belonging to a set from $\mathfrak{M}_W$.

2°. **Montel' s problem.** In connection with Montel' s problem, which consists in determining the structure of the set of points of non-uniform convergence of polynomials, M. A. Lavrentiev ([1]) introduced the classes of M- and M^*-sets and obtained a solution of this problem in terms of M-sets; he also pointed out the reduction of the problem of representing a function by a convergent sequence of polynomials to Baire' s problem, which consists in representing a function by a convergent and uniformly bounded sequence of polynomials. In other terms, the set of points of non-uniform convergence of polynomials was characterized by Hartogs and Rosenthal ([2]).

The lemma and the arguments establishing it lead to a new simple proof of these results of Lavrentiev, Hartogs, and Rosenthal, at the same time yielding a constructive characteristic of M- and M^*-sets of type F_σ.

Definition A. We shall say that E admits an A-representation if E is equal to the sum of expanding closed and bounded sets E_n, each of which is the boundary of some domain containing $z = \infty$.

Definition B. The set E admits a B-representation if E is equal to the sum of expanding closed and bounded sets, each of which is nowhere dense and does not separate the plane.

Definition C. The set E admits a C-representation through the sets E_n if E is equal to the sum of expanding closed bounded sets E_n, each of which does not separate the plane.

Theorem 1. In order that a set E of type F_σ be an M-set, it is necessary and sufficient that E admit an A-representation. In order that a set E of type F_σ be an M^*-set, it is necessary and sufficient that E admit a B-representation.

Let G be an arbitrary open set in the plane and E a set contained in G and closed relative to G. A point $\xi \in G$ is called an **irregular point** of a sequence of polynomials $P_n(z)$ converging in G if there is no neighborhood of ξ in which $P_n(z)$ converge uniformly.

From Theorem 1 there follows the following formulation of the aforementioned results of M. A. Lavrentiev.

Theorem 2. In order that E coincide with the set of all irregular points of some sequence of polynomials converging in G, it is necessary and sufficient that the following conditions be satisfied:

1. There exists an unbounded continuum containing E and belonging to the sum E and to the boundary of G.
2. The closed set E admits an A-representation.

Let $f(z)$ be defined and single-valued in G, and let E be the set of points of G at which $f(z)$ is non-analytic.

Theorem 3. In order that there exist a sequence of polynomials converging in G to $f(z)$, it is necessary and sufficient that the following conditions be fulfilled:

1. The set E admits an A-representation through sets E_n, $n \geq 1$.
2. For each E_n, $n \geq 1$, there exists a sequence of polynomials, uniformly bounded and converging on $\widetilde{E}_n$ to $f(z)$, where $\widetilde{E}_n = C\Delta_\infty(E_n)$, and $\Delta_\infty(E_n)$ is the domain complementary to E_n and containing $z = \infty$.

Theorem 4. For every function of the first Baire class on a set E of type F_σ to be the limit of a sequence of polynomials converging on E, it is necessary and sufficient that E admit a B-representation, i.e. be equal to the sum of closed sets on each of which every continuous function admits uniform approximation by polynomials.

3°. **The problem of the best majorant.** Let $M(z)$, $1 \leq M(z) \leq \infty$, be an arbitrary measurable function defined on the whole plane, and let $m(z) = \sup|P(z)|$ in the class of all polynomials satisfying the inequality $|P(z)| \leq (1+|z|)M(z)$ at every point of the plane. The problem of the best majorant, consisting in determining or estimating $m(z)$ in terms of the values of $M(z)$, is closely connected with questions of approximation and has been solved for only a few special classes of functions $M(z)$. The lemma makes it possible to give a general topological characterization of the best majorant $m(z)$, i.e. to indicate the structure of the sets on which $m(z)$ is finite and bounded.

Theorem 5. For there to exist a function $M(z)$, $1 \leq M(z) \leq \infty$, such that at all points of E, $m(z) < \infty$, and everywhere outside E, $m(z) = \infty$, it is necessary and sufficient that E be of type F_σ and admit a C-representation by means of certain sets E_n.

Let G denote the set of all points that are interior points of at least one of the E_n, $n \geq 1$, and let D be the totality of interior points of E. The set G is determined by a C-representation of E, and always $G \in D$. If E admits a C-representation, then such a representation is not unique; accordingly, for sets E satisfying the condition of the preceding theorem, different open sets G contained in D are obtained. Each such G is essentially connected with the property of the best majorant; this property is characterized by the proposition given below, which strengthens Theorem 5.

Theorem 6. For there to exist a function $M(z)$, $1 \leq M(z) \leq \infty$, such that $m(z)$, equal to $+\infty$ everywhere outside E, is finite at the points of E and bounded in a neighborhood of every point of $G \in E$, it is necessary and sufficient that the set E be equal to the sum of increasing closed bounded sets E_n, not separating the plane, and that G coincide with the totality of interior points of all E_n, $n \geq 1$.

For closed and nowhere dense sets E, the existence of a best majorant finite at the points of E and infinite on the complement of E is equivalent to the fact that the set E admits a B-representation, i.e. is an M^*-set. Accordingly, one can assert that the existence of a best majorant finite at the points of E and unbounded in every neighborhood of every point of E is equivalent to the fact that E admits an A-representation, i.e. is an M-set. In the first case the set of points where $m(z) = \infty$ coincides with the complement of E and is therefore everywhere dense on E; in the second case this set belongs to the complement of E and is likewise everywhere dense on E.

4°. **Weighted approximations by polynomials.** We shall say that the system of polynomials is **complete with weight** $h(z) > 0$ on a closed set E in the space C, if, whatever continuous function $f(z)$ on E with the condition $\lim_{z\to\infty} f(z) = 0$ and $\varepsilon > 0$ may be, there exists a polynomial $P(z)$ satisfying the inequality

$$\sup_{z\in E} h(z)|f(z) - P(z)| < \varepsilon.$$

Theorem 7. For there to exist on a set E of type F_σ a positive function $h(z)$ ensuring the weighted completeness of the system of polynomials on E, it is necessary and sufficient that E admit a B-representation.

Thus, for nowhere dense sets E of type F_σ, the possibility of representing on E any function that is pointwise discontinuous on perfect sub-

sets E, a function by means of a convergent sequence of polynomials is equivalent to the fact that the system of polynomials is complete on E in C with some positive weight $h(z)$.

The following proposition establishes the connection between weighted completeness and the property of the best majorant.

Theorem 8. For completeness of the system of polynomials with weight $h(z)$ on a closed nowhere dense set E, it is necessary and sufficient that the best majorant $m(z)$, corresponding to the function $M(z) \equiv 1/h(z)$, $z \in E$, $M(z) = \infty$, $z \notin E$, satisfy the condition $m(z) = \infty$ at every point outside E.

5°. Completeness of the system of rational functions.

Theorem 9. If a closed set E is the union of a countable number of closed sets E_n, on each of which the system of rational functions is complete in C, then completeness in C of the system of rational functions will also hold on E.

Corollary. In order that every continuous function on a closed set E admit uniform approximation by rational functions, it is necessary and sufficient that every closed subset of E contain a portion on which the system of rational functions is complete.

Theorem 10. On every closed set admitting an A-representation, the system of rational functions is complete.

Let us note that the indicated sufficient condition for completeness is formulated in topological terms.

Theorem 11. In order that every function of the first Baire class on a closed set E be the limit of a sequence of rational functions converging on E, it is necessary and sufficient that on E the system of rational functions be complete in the space C.

Let us note that the concept of W-reducibility for M- and M^*-sets without the lemma indicated above was considered by Hartogs and Rosenthal, who established Corollary 2 for these sets.

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CITED LITERATURE

1. M. Lavrentieff, *Acta. Sci. Ind.*, No. 441, Paris, (1936).

2. F. Hartogs, A. Rosenthal, *Math. Ann.*, 100, 212 (1928).

Note: Figure translations are in progress. See original paper for figures.