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Abstract

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ON SOME CONCEPTS RELATED TO THE NOTION OF THE GENUS OF A FIBERED SPACE

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We shall use the definitions and notation of the notes ($^{1-3}$). Let G be a topological group. We regard the space of the group G as a G-space, assuming that the group G acts on it by left translations. The join $G * \cdots * G$ of k copies of the G-space G will be denoted by A_k, and the orbit space $(A_k)_G$ of the group G in A_k by B_k. By the cone over a G-space X we shall mean the G-space $X * G$.

The genus of a G-space X (i.e., the genus of the fibration $X \to X_G$) will be denoted by $g(X)$; by virtue of Theorem 5 of note (^{1}), the genus of the G-space X can be defined as the least k for which there exists an admissible mapping of the G-space X into the G-space A_k. The homological genus of the G-space X will be denoted by $h(X)$.

It is easy to see that the genus of the cone $X * G$ over the G-space X is either equal to the genus of the G-space X, or exceeds it by one:

$$g(X) \leq g(X * G) \leq g(X) + 1.$$

This remark suggests the following.

Definition 1. The S-**genus** of a G-space X (denoted by $Sg(X)$) will be called the least of the numbers k for which

$$g(X * A_l) \leq k + l$$

for all sufficiently large l.

It is obvious that $Sg(X) \leq g(X)$. It can be proved that in the case when the group G is finite,

$$h(X * G) = h(X) + 1$$

and, consequently, in this case

$$h(X) \leq Sg(X).$$

Let us note the following proposition:

If $f : X \to Y$ is an admissible mapping of the G-space X into the G-space Y and $Sg(X) = Sg(Y)$, then the mapping f is not homotopic to zero.

If the group G is finite, then in this assertion the condition $Sg(X) = Sg(Y)$ can be replaced by the condition $h(X) = h(Y)$.

M. A. Krasnosel' skii posed in the book (^{5}) a question which can be expressed in the notation adopted here as follows: if $G = Z_2$ is the group of order two, then for every G-space X does the equality $Sg(X) = g(X)$ hold (in other words, is it always true that $g(X * Z_2) = g(X) + 1$)? We give an example which yields a negative answer to this question.

To each mapping $\varphi : S^m \to S^n$ of the m-dimensional sphere into the n-dimensional sphere, let us associate a Z_2-space E_φ by means of the following construction: the space E_φ is obtained from the sphere S^n by attaching two $(m+1)$-dimensional balls, the first of which is attached by means of the mapping φ, and the second by means of the mapping $T\varphi$ (T is the central symmetry of the sphere S^n); the unique nontrivial element of the group Z_2 acts on the sphere $S^n \subset E_\varphi$ as the central symmetry T, and interchanges the attached balls. If $\alpha \in \pi_m(S^n)$, $\varphi : S^m \to S^n$ is a mapping of the class α, then the genus of the Z_2-space E_φ does not depend on the choice of the mapping φ, and therefore we may introduce the notation

$$g(\alpha) = g(E_\varphi).$$

The following assertion holds:

If $\alpha \in \pi_m(S^n)$, where $n \leqslant 3$, $n = 7$, or $m < 2n - 1$, then $g(\alpha) = n + 1$ in the case when the element α has odd order, and $g(\alpha) = n + 2$ otherwise.

It is not difficult to verify that $E_\varphi * Z_2 = ES\varphi$, where $S\varphi$ is the suspension over the mapping φ; therefore, in order to refute Krasnosel' skii' s conjecture, it is enough to construct an element $\alpha \in \pi_m(S^n)$ such that $g(S\alpha) \neq g(\alpha) + 1$. But from the proposition formulated above it follows that for any element $\alpha \in \pi_3(S^2)$ having even Hopf invariant, one has $g(\alpha) = g(S\alpha) = 4$.

Definition 2. The **dual genus** of a G-space X (denoted by $\tilde{g}(X)$) is the greatest of the numbers k for which there exists an admissible mapping of the G-space A_k into X.

Definition 3. The **dual homological genus** of a G-space X (denoted by $\tilde{h}(X)$) is the least of the numbers k for which there is a cohomology class z of the classifying space B_G of the group G (with coefficients in a local system) satisfying the conditions $\varphi^*(z) = 0$, $\varphi^*_{k+1}(z) \neq 0$ (here φ, φ_{k+1} denote the characteristic mappings X_G into B_G and $B_{k+1} = (A_{k+1})_G$ into B_G). If the group G is discrete, then the condition $\varphi^*_{k+1}(z) \neq 0$ appearing in this definition may be replaced by the condition $\dim z \leqslant k$.

It is easy to see that $\tilde{g}(X) \leqslant \tilde{h}(X)$.

Let X be a G-space, and let $B \to B_G$ be the universal fibration with group G. Denote by X'_G the trajectory space $(X \times B)_G$ of the group G, acting coordinatewise in $X \times B$; this space is weakly homotopy equivalent to the space X_G. The projection $X \times B \to B$ gives rise to a mapping $p : X'_G = (X \times B)_G \to B_G$. The mapping p is a fibration with fibre X; this fibration will be denoted by $\mathfrak{B}(X'_G, B_G, X, p)$ (see (4), p. 209).

The problem of computing the dual genus can be reduced to the problem of the possibility of constructing a section by means of the following theorem:

Let X be a G-polyhedron, G a Lie group (not necessarily connected), and let $\mathfrak{B}'(Z, B_k, X, p')$ be the fibration over B_k induced from the fibration $\mathfrak{B}$ by means of the mapping $\varphi_k : B_k \to B_G$ (the characteristic mapping of the fibration $A_k \to B_k$). Then $\tilde{g}(X) \geqslant k$ if and only if the fibration $\mathfrak{B}'$ has a section.

Applying the theorem on the first obstruction to the extension of a section, we obtain the following assertion:

Let the space of the group G be a d-dimensional polyhedron, and let X be a G-polyhedron that is aspherical in dimensions $< s$ $(s \geqslant 2)$. If $s \geqslant (d+1)(k-1)$, then $\tilde{g}(X) \geqslant k$; if $s = (d+1)(k-1) - 1$, then $\tilde{g}(X) = k-1$ if and only if $\tilde{h}(X) = k-1$.

Let G be a discrete group, $G = S^1$ or $G = S^3$ (in these cases the space of the group G is aspherical in dimensions $< d$, where $d = \dim G$ is the dimension of this space). Then the dual genus of the G-polyhedron $X \geqslant k$ if and only if, over the $(d+1)(k-1)$-dimensional skeleton of the base B_G of the fibration $\mathfrak{B}(X'_G, B_G, X, p)$, there exists a section.

Using this assertion and applying known results on the second obstruction to the extension of a section (6,7), one can give conditions, expressed in terms of the homological properties of the spaces X and X_G, for the equality $\tilde{g}(X) = k-1$ to hold, if the space X is aspherical in dimensions

$$< s = (d+1)(k-1) - 2.$$

We give here only the result for $G = S^3$, which has an especially simple form.

Let $G = S^3$; let X be a G-space aspherical in dimensions

$$< s = 4(k-1) - 2;$$

b an element of the group $H^s(X_G, \pi_s(X))$ which maps

under the homomorphism $\pi^* : H^s(X_G, \pi_s(X)) \to H^s(X, \pi_s(X))$, generated by the identification map $\pi : X \to X_G$, into the fundamental class of the space

X. Then $\bar{g}(X) = k - 1$ if and only if the class $Sq^2 b \in H^{s+2}(X_G, \pi_{s+1}(X))$ is nonzero.

Definition 4. The **dual** S**-genus** of a G-space X (denoted $S\bar{g}(X)$) is the largest of the numbers k for which $\bar{g}(X * A_l) \geq k + l$ for some l.

It is obvious that $S\bar{g}(X) \geq \bar{g}(X)$. If G is a finite group, then one can show that $S\bar{g}(X) \leq \bar{h}(X)$.

Let us note that the S-genus and the dual S-genus can be simply characterized by means of the notion of an R-mapping: if R is a G-space into which every G-space of genus $\leq g(R)$ can be admissibly mapped (for example, if $R = G$), then $Sg(X) \leq k$ $[S\bar{g}(X) \geq k]$ if and only if there exists an R-mapping of the G-space X into the G-space A_k (of the G-space A_k into the G-space X).

The S-genus and the dual S-genus do not decrease under R-mappings; more precisely, if the G-space R satisfies the conditions imposed above and there exists an R-mapping of the G-space X into the G-space Y, then $Sg(X) \leq Sg(Y)$ and $S\bar{g}(X) \leq S\bar{g}(Y)$. If the group G is finite, then under the same conditions $h(X) \leq h(Y)$ and $\bar{h}(X) \leq \bar{h}(Y)$.

Let G be a finite group; let R be an r-dimensional G-polyhedron homotopy equivalent to the r-dimensional sphere (such a G-polyhedron exists if and only if every abelian subgroup of the group G is cyclic ([8])).

Suppose that X is a G-polyhedron; $Y = D_{nX}$ is the G-polyhedron n-dual to the G-polyhedron X with respect to R. Then:

1. $Sg(X) + S\bar{g}(Y) = (r+1)n$.
2. $h(X) + \bar{h}(Y) = (r+1)n$.
3. If $\dim X \leq 2Sg(X) - 3$, then $g(X) = Sg(X)$.
4. If the polyhedron X is aspherical in dimensions $< \frac{1}{2} S\bar{g}(X)$, then $\bar{g}(X) = S\bar{g}(X)$.

Analogous assertions can be proved in the case when $G = S^1$ or $G = S^3$.

As is clear from the theorems formulated above, the dual genus is easier to compute than the genus of a G-space; therefore the relation

$$Sg(X) = (r+1)n - S\bar{g}(D_{nX})$$

may perhaps be used to compute the genus of concrete G-polyhedra.

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