MATHEMATICS

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Abstract

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ON THE STABILITY OF AN EQUILIBRIUM POSITION OF A HAMILTONIAN SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS IN THE GENERAL ELLIPTIC CASE

(Presented by Academician A. N. Kolmogorov, 9 XII 1960)

§ 1. Let the point $p = q = 0$ be a stationary point of the system

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}, \tag{1}$$

where $H(p, q, t)$ is a function analytic in p, q, t, periodic in t with period 2π. The case is called **elliptic** when this equilibrium position is stable in the first (linear) approximation. Then, as Birkhoff showed (^{1}), for a suitable choice of the variables p, q, the Hamiltonian function has the form

$$H = \lambda r + c_2 r^2 + \cdots + c_{nr}^{n} + \widetilde{H}(p, q, t), \tag{2}$$

where $2r = p^2 + q^2$, $\widetilde{H} = O(r^{n+1})$ is an analytic function of p, q, t, and $n \geq 2$ is arbitrary. We shall call the case **general elliptic** when among the constants c_l $(2 \leq l < \infty)$ there is one different from zero.

§ 2. Examples are known in which the equilibrium position is unstable and λ is rational (^{2}). We shall consider the case of irrational λ. Denote by Λ_K the set of such λ for which the inequalities

$$|\lambda n - m| > \frac{K}{(|m| + |n|)^2} \tag{3}$$

are satisfied for all integers $m, n > 0$. Denote by Λ the union of the points of density of all sets Λ_K. As is known, the complement of Λ on the line has measure zero (^{3}).

Theorem 1. If $\lambda \in \Lambda$, then the equilibrium position $0, 0$ of the system of equations (1) with Hamiltonian function $H(p, q, t)$ of general elliptic type (2) is stable.

Theorem 2. Under the hypotheses of Theorem 1, in any neighborhood of the circle $p = q = 0$ of the space p, q, t there exists an analytic invariant torus T_μ with equation $r = r(\varphi, t)$ $\left(\varphi = \operatorname{arctg} \frac{p}{q}\right)$. On the torus T_μ one can introduce an analytic coordinate $\psi(\varphi, t)$ such that equations (1) on the torus T_μ take the form $\dot{\psi} = \mu$. The set formed by the tori T_μ has positive measure in the space p, q, t.

Theorem 3. Let the Hamiltonian function have the form

$$H(r, \varphi, t) = H_0(r) + \widetilde{H}(r, \varphi, t), \tag{4}$$

where $dH_0/dr = \mu + \Omega(r)$, $\mu \in \Lambda_K$, $\Omega(0) = 0$, and the function

$$\widetilde{H} = \sum_{m^2+n^2 \neq 0} H_{mn}(r) e^{i(m\varphi + nt)}$$

for $|\operatorname{Im} \varphi, t| \leq \rho,\ |r| \leq \rho_r = \delta^k$ is analytic and satisfies the inequality

$$|\widetilde{H}| \leq M = \delta^N, \tag{5}$$

and the function $\Omega(r)$, for $|r| \leqslant \rho_r$, is analytic and

$$\delta^a = \theta \leqslant \left|\frac{d\Omega}{dr}\right| \leqslant \Theta = \delta^{-b}.$$

Here $\delta > 0$ is a certain constant; N, k, a, b are natural numbers. If the inequalities

$$\begin{gathered} 2k + 28 + 2a + 4b < N < 3k - 14 - 2b; \\ \delta < 10^{-6} K^2; \quad \delta < 0.1\,\rho, \end{gathered} \tag{7}$$

are satisfied, then there exist functions $R(\varphi, t)$, $\Psi(\varphi, t)$, of period 2π in φ and t, analytic for $|\operatorname{Im} \varphi, t| \leqslant 0.1\,\rho$, and such that on the torus $r = R(\varphi, t)$, from the equations

$$\dot{\varphi} = \frac{\partial H}{\partial r}, \qquad \dot{r} = -\frac{\partial H}{\partial \varphi}$$

it follows that $\dot{\psi} = \mu$ (here $\psi = \varphi + \Psi$).

Theorem 1 follows from Theorem 2, since the tori T_μ separate the circle $r = 0$ from the remaining part of the space p, q, t. Theorem 2 follows from Theorem 3: it is not difficult to see that, under the hypotheses of Theorem 2, there exist arbitrarily small toroidal rings $|r - r_0| \leqslant \rho_r$ around the circle $r = 0$ to which Theorem 3 is applicable, if in it one takes $r - r_0$ as the variable r.

§ 3. The last two theorems are generalized to systems with n degrees of freedom. However, the resulting invariant $(n+1)$-dimensional tori do not divide the $(2n+1)$-dimensional phase space p, q, t, and the question of stability remains open. Analogous theorems can also be proved concerning a neighborhood of the equilibrium position of an autonomous Hamiltonian system. In this case, in the $(2n-1)$-dimensional manifolds $H(p, q) = h$ there lie tori of dimension n. Hence it follows:

Theorem 4. *The equilibrium position of an autonomous Hamiltonian system of equations with two degrees of freedom in the general elliptic case is stable, if* $\lambda_2/\lambda_1 \in \Lambda$.

By the general elliptic case we mean here the case when the analytic function H, in suitable coordinates, has the form (^{1})

$$H(p_1, p_2, q_1, q_2) = \lambda_1 r_1 + \lambda_2 r_2 + H_0(r_1, r_2) + \widetilde{H}(p_1, p_2, q_1, q_2),$$

where

$$H_0(r_1, r_2) = \sum_{i+j=2}^{n} c_{ij} r_1^i r_2^j, \qquad \widetilde{H} = O(r_1 + r_2)^{n+1}, \qquad 2r_i = p_i^2 + q_i^2$$

and $h(\varepsilon) = H_0(\varepsilon\lambda_2, -\varepsilon\lambda_1)$ does not vanish identically.

One can also show that any analytic canonical mapping of the plane onto itself near a fixed point of general elliptic type is stable, if its rotation number $\lambda \in \Lambda$. Theorems 2 and 3 admit a corresponding generalization even in the multidimensional case.

§ 4. We shall now outline the proof of Theorem 3. It is a strengthening of A. N. Kolmogorov' s theorem on the preservation of conditionally periodic motions under a small change of the Hamiltonian function (^{4}). The invariant torus is found, as in A. N. Kolmogorov' s work, by successive approximations of Newton-method type. This method gives such rapid convergence that it cannot be destroyed by the small divisors appearing in formula (9).

Basic Lemma. *Under assumptions (4)–(7) of Theorem 3, there exist an analytic function*

$$\widetilde{F}(\bar{r}, \varphi, t) = \sum_{m^2+n^2 \neq 0} F_{mn}(\bar{r}) e^{i(m\varphi + nt)}$$

and a number $\bar{r}^*$ *such that the canonical transformation*

$$\bar{\varphi} = \varphi + \partial\widetilde{F}/\partial\bar{r}, \qquad \bar{r} = \bar{r}' - \bar{r}^*, \qquad r = \bar{r}' + \partial\widetilde{F}/\partial\varphi$$

brings the Hamiltonian function (4) to the form $\bar{H}(\bar{r}, \bar{\varphi}, t) = \bar{H}_0 + \tilde{\bar{H}}$, where $d\bar{H}_0/d\bar{r} = \mu + \bar{\Omega}(\bar{r})$, $\bar{\Omega}(0) = 0$, and the function

$$\tilde{\bar{H}}(\bar{r}, \bar{\varphi}, t) = \sum_{m^2+n^2 \neq 0} \bar{H}_{mn}(\bar{r}) e^{i(m\bar{\varphi}+nt)}$$

is analytic for $|\operatorname{Im}\bar{\varphi}, t| \leq \bar{\rho} = \rho - 3\delta$, $|\bar{r}| \leq \bar{\rho}_r = \delta^k$, and satisfies the inequality

$$|\tilde{\bar{H}}| \leq \bar{M} = \delta^N,$$

while $\bar{\Omega}(\bar{r})$ is analytic for $|\bar{r}| \leq \bar{\rho}_r$ and

$$\delta^a = \bar{\theta} \leq \left|\frac{d\bar{\Omega}}{d\bar{r}}\right| \leq \bar{\Theta} = \delta^{-b}.$$

In these formulas $\bar{\delta} = \delta^{1^{1}/_{2}}$.

Theorem 3 is derived from the fundamental lemma without special difficulty, since the error of the s-th approximation M_s is not greater than $M^{(1^{1}/_{2})^s}$.

Not being able to give here the proof of the fundamental lemma, I shall indicate only the method of constructing $\tilde{F}$ and $\bar{r}^*$. As is known, $\bar{H}(\bar{r}, \bar{\varphi}, t) = H'(\bar{r}', \varphi, t) = \hat{H}(\bar{r}', \varphi, t)$, where

$$\hat{H}(\bar{r}', \varphi, t) = H(r(\bar{r}', \varphi, t), \varphi, t) + \frac{\partial\tilde{F}(\bar{r}', \varphi, t)}{\partial t}.$$

Obviously,

$$\hat{H}(\bar{r}', \varphi, t) = H_0(\bar{r}') + \hat{S}_1 + \hat{S}_2 + \hat{S}_3,$$

where

$$\hat{S}_1(\bar{r}', \varphi, t) = \mu\frac{\partial\tilde{F}}{\partial\varphi} + \frac{\partial\tilde{F}}{\partial t} + \tilde{H};$$

$$\hat{S}_2(\bar{r}', \varphi, t) = H_0(\bar{r}) - H_0(\bar{r}') - \mu(r - \bar{r}'), \qquad |\hat{S}_2| = |\Omega|\left|\frac{\partial\tilde{F}}{\partial\varphi}\right|; \tag{8}$$

$$\hat{S}_3(\bar{r}', \varphi, t) = \tilde{H}(\bar{r}) - \tilde{H}(\bar{r}'), \qquad |\hat{S}_3| = \left|\frac{\partial \tilde{H}}{\partial r}\right|\left|\frac{\partial \tilde{F}}{\partial \varphi}\right|.$$

The function $\tilde{F}$ is determined from the condition $\hat{S}_1 \equiv 0$:

$$F_{mn} = \frac{iH_{mn}}{\mu m + n}. \tag{9}$$

Passing to the variables $\bar{r}, \bar{\varphi}, t$, we find

$$\bar{H}'(\bar{r}, \bar{\varphi}, t) = \bar{H}_0'(\bar{r}) + \tilde{S}_{2_0}(\bar{r}) + \tilde{S}_{3_0}(\bar{r}) + \tilde{\tilde{H}}'(\bar{r}, \bar{\varphi}, t) = \tilde{\tilde{H}}_0 + \tilde{\tilde{H}}',$$

where $\tilde{\tilde{H}}'(\bar{r}, \bar{\varphi}, t) = \tilde{S}_2 + \tilde{S}_3$ combines the variable terms of the Fourier series in $\bar{\varphi}, t$ of the functions

$$S_i(\bar{r}, \bar{\varphi}, t) = \tilde{S}_i + S_{i_0}(\bar{r}) = \hat{S}_i(\bar{r}, \varphi(\bar{r}, \bar{\varphi}, t), t), \qquad (i = 2, 3).$$

Now $\bar{r}^*$ is determined from the equation

$$\left.\frac{d\tilde{\tilde{H}}_0}{d\bar{r}'}\right|_{\bar{r}^*} = \mu.$$

In this, to estimate $\bar{r}^*$ one uses inequality (6).

When inequalities (7) are satisfied, the quantity $\tilde{\tilde{H}}$, estimated by formulas (8), does not exceed $M^{1^1/_2} = \bar{M}$.

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