MATHEMATICS

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Abstract

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ON THE COMPLETION AND CLOSURE OF AN INCOMPLETE SYSTEM OF FUNCTIONS $\{e^{-\mu_k x} x^{s_k - 1}\}$

1°. Let $\{\mu_k\}$, $\operatorname{Re}\mu_k > 0$ $(k = 1, 2, \dots)$, be an arbitrary sequence of complex numbers, among which numbers may occur (but not necessarily consecutively) with finite or even infinite multiplicity. Denoting by $s_k \geq 1$ the multiplicity of the number μ_k among the system $\{\mu_1, \dots, \mu_k\}$, we associate with the sequence of numbers $\{\mu_k\}$ $(k = 1, 2, \dots)$ the sequence of functions $\{e^{-\mu_k x} x^{s_k - 1}\}$ $(k = 1, 2, \dots)$ from $L_2(0, +\infty)$.

In the space of functions $L_2(0, +\infty)$ the well-known approximation theorem of Müntz can be formulated as follows*:

For completeness of the system of functions

$$\{e^{-\mu_k x} x^{s_k - 1}\} \qquad (k = 1, 2, \dots) \tag{1}$$

in $L_2(0, \infty)$ it is necessary and sufficient that the condition

$$\sum_{k=1}^{\infty} \frac{\operatorname{Re}\mu_k}{1 + |\mu_k|^2} = +\infty \tag{2}$$

be satisfied.

Let us agree that $\{\mu_k\} \subset S$, according as the series (2) diverges or converges. We shall denote further by $M_2^{(n)}\{\mu_k\}$ the linear span of the system of functions $\{e^{-\mu_k x} x^{s_k - 1}\}$ $(k = 1, 2, \dots, n \leq \infty)$ in the metric of $L_2(0, \infty)$. By Müntz' s theorem, $M_2^{(\infty)}\{\mu_k\} \equiv L_2(0, +\infty)$ only if $\{\mu_k\} \subset R$. In this connection, apparently long ago the problem was posed of a complete characterization of the whole class $M_2^{(\infty)}\{\mu_k\}$ (or of the class $C^{(\infty)}\{\mu_k\}$ when taking the closure of the same system in the metric of uniform approximation) in the case when $\{\mu_k\} \subset S$.

We note that some necessary, but by no means sufficient, conditions which the functions of these classes must satisfy, in the particular assumptions $\operatorname{Im}\mu_k = 0$; $s_k = 1$ $(k = 1, 2, \dots)$ on the sequence $\{\mu_k\} \subset S$, were established by Laurent Schwartz ([6]) and independently by A. F. Leont' ev ([7]).

In the present article we give formulations of several new results on the completion and biorthogonalization of the incomplete system (1) and on the complete characterization of the class $M_2^{(\infty)}\{\mu_k\}$ for $\{\mu_k\} \subset S$. In obtaining these results we rely essentially on the concept of a unitary pair of operators and on its analytic characteristic, given in the note ([8]).

2°. Putting, for $n = 1, 2, ...,$

$$B_n(z) = \prod_{k=1}^{n} \frac{z+\mu_k}{z-\bar{\mu}_k}\,\varepsilon_k, \qquad \varepsilon_k = \frac{|(1-\mu_k)(1+\mu_k)|}{(1-\mu_k)(1+\mu_k)}$$

$(k = 1, 2, ...)$, in the case $\{\mu_k\} \subset S$, we shall also include here the Blaschke products

$$B_\infty(z) \equiv B(z) \equiv \prod_{k=1}^{\infty} \frac{z+\mu_k}{z-\bar{\mu}_k}\,\varepsilon_k,$$

which converge everywhere except at the points of the imaginary axis.

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* In the case $s_k = 1$ $(k = 1, 2, ...)$ see ([1],[3]), and also the book ([3]). In the case $s_k \geq 1$, but with $\operatorname{Im}\mu_k = 0$, see ([4],[5]). The proof of the theorem in the general formulation given here is in fact contained in the moment method of proof used in the book ([3]), under the condition $s_k = 1$ $(k = 1, 2, ...)$, which is completely superfluous.

$\operatorname{Re} z = 0$ and the points of the sequence $\{\bar{u}_k\}$. Observing further that for any measurable function $\varphi(x)$, bounded on the whole axis $(-\infty, +\infty)$, for each fixed value $\xi \in (-\infty, +\infty)$ there exists the mean limit on $-\infty < x < +\infty$

$$L(\xi;\varphi(x)) \equiv \frac{1}{2\pi}\,\text{l.i.m.}_{\sigma\to\infty} \int_{-\sigma}^{\sigma} \varphi(\tau)\,\frac{e^{-i\xi\tau}-1}{-i\tau}\,e^{ix\tau}\,d\tau \in L_2(-\infty,+\infty), \qquad (3)$$

we introduce for consideration the functions $K_n(\xi, x) \equiv L(\xi; B_n(ix))$, $K_n^*(\xi, x) \equiv L(\xi; \overline{B_n(ix)})$, and, under the condition $\{\mu_k\} \subset S$, also the functions $K_\infty(\xi, x) \equiv K(\xi, x) \equiv L(\xi, B(ix))$, $K_n^*(\xi, x) \equiv K^*(\xi, x) \equiv L(\xi, \overline{B(ix)})$, where by $B(ix)$ is meant the boundary value of the Blaschke function on the imaginary axis, which exists and is of modulus one almost everywhere for all $-\infty < x < +\infty$.

Denote by $\{\gamma_k(x)\}$ $(k = 1, 2, ...)$ the orthogonalization of the sequence of functions $\{e^{-\mu_k x} x^{k-1}\}$ $(k = 1, 2, ...)$ on the half-axis $(0, +\infty)$. By a direct calculation one easily verifies the validity of the integral formula

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\sqrt{\frac{\operatorname{Re}\mu_k\,B_{k-1}(i\tau)}{\pi}}\,i\tau+\mu_k\,e^{ix\tau}\,d\tau=\begin{cases}\gamma_k(x), & 0<x<\infty,\\ 0, & -\infty<x<0,\end{cases}\tag{4}$$

(where one must put $B_0(i\tau)\equiv 1$), obtained in another way by G. V. Badalyan ([5]) for the case when $\operatorname{Im}\mu_k=0$. Finally, putting

$$R_n(\xi,x)\equiv\sum_{k=1}^{n}\left(\int_0^{\xi}\gamma_k(t)\,dt\right)\gamma_k(x)\quad(n=1,2,...),$$

we note that for each fixed value $\xi\in(0,+\infty)$ on the half-axis $0<x<\infty$ there exists the mean limit

$$R_\infty(\xi,x)\equiv R(\xi,x)=\text{l.i.m.}_{n\to\infty}R_n(\xi,x).\tag{5}$$

The following important auxiliary theorem is valid.

Theorem 1. *For any $1\le n<\infty$, and also for $n=\infty$, if $\{\mu_k\}\subset S$, the following equations hold among the functions $K_n(\xi,x)$, $K_n^*(\xi,x)$, $R_n(\xi,x)$:*

$$\left.\begin{aligned}&\text{a)}\quad\int_0^{\infty}\overline{K_n(\xi,x)}\,K_n(\eta,x)\,dx+\int_0^{\infty}\overline{R_n(\xi,x)}\,R_n(\eta,x)\,dx\\ &\text{b)}\quad\int_0^{\infty}\overline{K_n^*(\xi,x)}\,K_n^*(\eta,x)\,dx\end{aligned}\right\}=\min(\xi,\eta);$$

$$\text{c)}\quad\int_0^{\eta}K_n(\xi,x)\,dx=\int_0^{\xi}\overline{K_n^*(\eta,x)}\,dx;$$

$$\text{d)}\quad\int_0^{\eta}R_n(\xi,x)\,dx=\int_0^{\xi}\overline{R_n(\eta,x)}\,dx,$$

where $\xi>0$, $\eta>0$ are arbitrary.

Hence, using Theorem 1 of our preceding note ([8]), we may assert that the quadruple of functions $K_n(\xi,x)$, $K_n^*(\xi,x)$, $R_n(\xi,x)\equiv R_n^*(\xi,x)$ generates a certain unitary pair of operators $\{U_1,U_2\}$, acting in the space $L_2(0,+\infty)$, in accordance with the following main theorem:

Theorem 2. *Let $f(x)\in L_2(0,+\infty)$ be arbitrary; then for any $1\le n<\infty$, and also for $n=\infty$, if $\{\mu_k\}\subset S$, the representation holds*

$$\int_0^\xi f(x)\,dx = \sum_{k=1}^n c_k(f) \int_0^\xi \gamma_k(x)\,dx + \int_0^\infty \overline{K_n(\xi,x)}\, g_n(x)\,dx, \qquad \xi \in (0,+\infty), \tag{6}$$

where $g_n(x) \in L_2(0,+\infty)$,

$$\int_0^\xi g_n(x)\,dx = \int_0^\infty \overline{K_n^*(\xi,x)}\, f(x)\,dx, \quad \xi \in (0,\infty)$$

and

$$c_k(f) = \int_0^\infty f(x)\overline{\gamma_k(x)}\,dx \quad (k = 1, 2, \ldots);$$

moreover always

$$\int_0^\infty |f(x)|^2\,dx \equiv \|f\|^2 = \sum_{k=1}^n |c_k(f)|^2 + \|g_n\|^2. \tag{7}$$

Thus, the representation (6) gives an effective completion of each finite system $\{e^{-\mu_k x} x^{s_k - 1}\}$ $(k = 1, 2, \ldots, n)$, as well as of the whole system (1) in the case when $\{\mu_k\} \subset S$, i.e. when this system is incomplete.

3°. Let us now note that, in general, under the condition $\{\mu_k\} \subset S$, Theorem 2 implies:

Corollary. The class $M_2^{(n)}\{\mu_k\}$ $(1 \le n \le \infty)$ coincides with the set of functions in $L_2(0,+\infty)$ satisfying the additional condition

$$\int_0^\infty \overline{K_n^*(\xi,x)}\, f(x)\,dx = 0, \quad \xi \in (0,+\infty). \tag{8}$$

To put condition (8) into a more transparent and equivalent form, we introduce the definition: a function $F(w) \in H_2^{(+)}$ (respectively $F(w) \in H_2^{(-)}$) if it is holomorphic in the half-plane $\operatorname{Re} w > 0$ $(\operatorname{Re} w < 0)$ and

$$\int_{-\infty}^{+\infty} |F(u+iv)|^2\,dv \le M_0 < +\infty$$

for all $0 < u < \infty$ $(-\infty < u < 0)$, where M_0 does not depend on u. From Theorem 2 it follows:

Theorem 3. If $\{\mu_k\} \subset S$, then the class $M_2^{(n)}\{\mu_k\}$ $(1 \le n \le \infty)$ coincides with the subset of those $f(x) \in L_2(0,+\infty)$ for which the expression

$$F(iv) = \text{l. i. m.}_{\sigma\to\infty} \int_0^\sigma e^{-ivx} f(x)\, dx \in L_2(-\infty, +\infty) \tag{9}$$

(being, as is known, the boundary function for some $F(w) \in H_2^{(+)}$) is such that the product $F(iv)B_n(iv)$ is the boundary function for a certain function $F_0(w) \in H_2^{(-)}$.

From this it follows immediately:

Corollary*. If $\{\mu_k\} \subset S$, and moreover $\lim |\mu_k| = \infty$, then the class $M_2^{(n)}\{\mu_k\}$ $(1 \le n \le \infty)$ coincides with the set of functions $f(x)$ represented in the form

$$f(x) = \text{l. i. m.}_{\sigma\to\infty} \frac{1}{2\pi} \int_{-\sigma}^\sigma F(iv) e^{ivx}\, dv, \quad 0 < x < \infty, \tag{10}$$

where $F(w)$ is an arbitrary meromorphic function in the entire w-plane satisfying the conditions $F(w) \in H_2^{(+)}$, $F(w)B_n(w) \in H_2^{(-)}$.

When additional restrictions are imposed on the sequence $\{\mu_k\}$, one can indicate necessary conditions for a function to belong to the class $M_2^{(\infty)}\{\mu_k\}$.

We shall say that $\{\mu_k\} \subset S(\alpha)$ $(0 \le \alpha < 1)$, if $\{\mu_k\} \subset S$, and, in addition, $|\arg \mu_k| \le \frac{1}{2}\pi\alpha$ $(0 \le \alpha < 1)$, $k = 1, 2, \ldots$. Then the following holds:

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* In the statement of Theorem 3 and its corollary we have included also the cases when $1 \le n < +\infty$, in order to emphasize that, in general, for all $1 \le n \le +\infty$ the result has a unified formulation, although for finite n it is established very simply.

Theorem 4. a) If $\{\mu_k\} \subset S(\alpha)$ $(0 \le \alpha < 1)$, then every function $f(x) \in M_2^{(\infty)}\{\mu_k\}$ is holomorphic in the angle $\Delta_\alpha : |\arg z| < \frac{1}{2}\pi(1-\alpha)$ and, moreover,

$$\int_0^\infty |f(re^{i\psi})|^2\, dr < +\infty, \qquad |\psi| < \frac{1}{2}\pi(1-\alpha). \tag{11}$$

b) If $\{\mu_k\} \subset S(\alpha)$ $(0 \le \alpha < 1)$, and $\lim |\mu_k| = \infty$, then for every function $f(x) \in M_2^\infty\{\mu_k\}$ the expansion

$$f(z) = \sum_{k=1}^\infty c_k(f)\gamma_k(z), \qquad z \in \Delta_\alpha, \tag{12}$$

holds, converging uniformly and absolutely in every subdomain of Δ_α.

4°. Let us now note that under the condition $\{\mu_k\} \subset S$, for every $n \geq 1$ the multiplicity p_n of the number μ_n in the entire sequence $\{\mu_k\}$ is always finite, and it is obvious that $1 \leq s_n \leq p_n$ $(n = 1, 2, ...)$. Finally denoting

$$\omega_n(x) = \frac{1}{2\pi} \underset{\sigma \to \infty}{\mathrm{l.\,i.\,m.}} \int_{-\sigma}^{\sigma} \left\{ \frac{p!}{(s_n - 1)!\, B^{(p_n)}(-\mu_n)} \frac{B(it)}{(it + \mu_n)^{p_n - s_n + 1}} \right\} e^{itx}\, dt, \quad (13)$$

we have:

Theorem 5. a) If $\{\mu_k\} \subset S$, then the systems of functions $\{e^{-\mu_k x} x^{s_k - 1}\}_1^\infty$, $\{\omega_k(x)\}_1^\infty$ are biorthogonal on the half-axis $(0, +\infty)$. Moreover, if $f(x) \in M_2^{(\infty)}\{\mu_k\}$ and

$$S_n(x, f) = \sum_{k=1}^{n} c_k(f)\gamma_k(x) = \sum_{k=1}^{n} a_k^{(n)}(f) e^{-\mu_k x} x^{s_k - 1}, \quad (14)$$

then all the limits exist

$$a_k(f) = \lim_{n \to \infty} a_k^{(n)}(f) = \int_0^\infty f(x)\overline{\omega_k(x)}\, dx \qquad (k = 1, 2, ...). \quad (15)$$

b) If $\{\mu_k\} \subset S(\alpha)$ $(0 \leq \alpha < 1)$, $\lim |\mu_k| = \infty$, then for every function $f(x) \in M_2^{(\infty)}\{\mu_k\}$ the formal expansion

$$f(z) \sim \sum_{k=1}^{\infty} a_k(f) e^{-\mu_k z} z^{s_k - 1}, \qquad a_k(f) = \int_0^\infty f(x)\overline{\omega_k(x)}\, dx, \quad (16)$$

at least after a suitable rearrangement of its terms, will converge uniformly to $f(z)$ inside the domain Δ_α.

In conclusion, we note that in the works of L. Schwartz and A. F. Leont' ev[6,7], by entirely different methods, results were established which in essence differ little from assertion a) of Theorem 4 and from Theorem 5 (without the fact of the existence of a biorthogonal system), but only for the case when $\{\mu_k\} \subset S(0)$ (i.e. when $\mathrm{Im}\, \mu_k = 0$, $\sum |\mu_k|^{-1} < +\infty$) and $s_k = 1$ $(k = 1, 2, ...)$.

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Note: Figure translations are in progress. See original paper for figures.