# Mathematics

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## Abstract

## Full Text

*Mathematics*

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# RIEMANNIAN SPACES WITH ORTHOGONAL AND SYMPLECTIC GROUPS OF MOTIONS AND AN IRREDUCIBLE GROUP OF ROTATIONS

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1°. In the present note we describe homogeneous nonsymmetric Riemannian spaces $M = \mathfrak{G}/\mathfrak{H}$ with stationary subgroup $\mathfrak{H}$, acting irreducibly on the tangent space; $\mathfrak{G}$ is a connected, $\mathfrak{H}$ a connected compact Lie group. In ([3]) such spaces were found in the special case $\mathfrak{G} = SU$ (the unitary unimodular group). Let now $\mathfrak{G}$ be a compact simple Lie group of type $B_n, C_n, D_n$ (if $\mathfrak{G}$ is not simple or is noncompact, the required spaces do not exist).

The problem posed reduces to the following one (problem A, see ([3])):
Find all embeddings

$$\varphi(H^+) \subset G^+ \tag{1}$$

($\varphi(H^+)$ is the complex form of a compact linear representation $\varphi(H)$ of the Lie algebra of the group $\mathfrak{H}$; $G^+$ is the Lie algebra of all complex skew-symmetric matrices or the complex symplectic Lie algebra) such that the Kronecker product $\varphi(H^+) \times \widetilde{\varphi}(H^+)$ ($\widetilde{\varphi}$ is the representation contragredient to $\varphi$; of course, $\widetilde{\varphi}$ is equivalent to $\varphi$) in the space $G^+$ decomposes, over the adjoint representation on $H^+ \subset G^+$, only into two mutually contragredient irreducible representations or into a single irreducible representation.

The product $\varphi(H^+) \times \widetilde{\varphi}(H^+)$ acts, by definition, in the space of all matrices $g$, in particular in the space $G^+$, by the formula

$$\varphi \times \widetilde{\varphi}(h)\, g = \varphi(h)g + g\widetilde{\varphi}(h), \qquad h \in H^+. \tag{2}$$

Let $H^+$ first be a semisimple complex algebra.

2°. **Lemma 1.** Let $\varphi(H^+)$ be an irreducible linear representation of the semisimple algebra $H^+$ by skew-symmetric complex matrices, and let $\Lambda$ be its highest weight.

1. If $(\Lambda, \alpha) \neq 0$, where $\alpha$ is a simple root of $H^+$, then for the Kronecker product $\varphi(H^+) \times \widetilde{\varphi}(H^+)$, acting in the space of all skew-symmetric matrices $G^+$, there exists an invariant irreducible subspace with highest weight of the representation $2\Lambda - \alpha$.

2. If, moreover,

$$\frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \geq 3,$$

then in $G^+$ there is, in addition, an invariant irreducible subspace with highest weight of the representation $2\Lambda - 3\alpha$.

3. If $\alpha, \beta, \gamma$ are simple roots of $H^+$ such that

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = -1, \qquad \frac{2(\alpha, \gamma)}{(\alpha, \alpha)} = -1, \qquad \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} = 2, \qquad \frac{2(\Lambda, \beta)}{(\beta, \beta)} = 0, \qquad \frac{2(\Lambda, \gamma)}{(\gamma, \gamma)} = 0, \tag{3}$$

then in $G^+$ there is an invariant irreducible subspace with highest weight of the representation $2\Lambda - 3\alpha - \beta - \gamma$.

**Lemma 2.** Let $\varphi(H^+)$ be an irreducible linear representation of the semisimple algebra $H^+$ by complex symplectic matrices (here and below we shall call infinitesimal complex symplectic matrices the matrices of senior weight $\Lambda$).

**Table 1**

*SO(N)*

| No. | Type $H^+$ | Scheme $\varphi$ | $N$-dimension of $\varphi$ | Dimension of the homogeneous space |
|---|---|---|---|---|
| 1 | $A_m,\ m \geq 2$ | $A_m$ chain with end labels $1, \dots, 1$ | $m^2 + 2m$ | $\frac{m(m-1)(m+2)(m+3)}{2}$ |
| 2 | $A_3$ | $A_3$ chain with label 2 on the middle node | 20 | 175 |
| 3 | $A_1$ | single node with label 4 | 5 | 7 |

| No. | Type $H^+$ | Scheme $\varphi$ | $N$-dimension of $\varphi$ | Dimension of the homogeneous space |
|---|---|---|---|---|
| 4 | $B_m,\ m \geq 2$ | $B_m$ chain with label 2 at the first node | $m(2m+3)$ | $\frac{m(m+1)(2m-1)(2m+5)}{2}$ |
| 5 | $B_m,\ m \geq 3$ | $B_m$ chain with label 1 at the second node | $m(2m+1)$ | $\frac{m(m-1)(2m+3)(2m+1)}{2}$ |
| 6 | $B_3$ | $B_3$ scheme with label 1 at the terminal short-root node | 8 | 7 |
| 7 | $D_m,\ m \geq 4$ | $D_m$ forked scheme with label 2 at the first node | $(m+1)(2m-1)$ | $\frac{(m+2)(m-1)(2m+1)(2m-1)}{2}$ |
| 8 | $D_m,\ m \geq 4$ | $D_m$ forked scheme with label 1 at the second node | $m(2m-1)$ | $\frac{m(m+1)(2m-1)(2m+3)}{2}$ |
| 9 | $C_m,\ m \geq 2$ | $C_m$ chain with label 2 at the first node | $m(2m+1)$ | $\frac{m(m-1)(2m+1)(2m+3)}{2}$ |
| 10 | $C_m,\ m \geq 3$ | $C_m$ chain with label 1 at the second node | $(m-1)(2m+1)$ | $\frac{(m+1)(m-2)(2m+1)(2m-1)}{2}$ |

| No. | Type $H^+$ | Scheme $\varphi$ | $N$-dimension of $\varphi$ | Dimension of the homogeneous space |
|---|---|---|---|---|
| 11 | $C_3$ | $C_3$ chain with label 1 at the last node | 12 | 825 |
| 12 | $E_6$ | $E_6$ scheme with label 1 at the branching node | 78 | 2925 |
| 13 | $E_7$ | $E_7$ scheme | 133 | 8645 |
| 14 | $E_8$ | $E_8$ scheme with label 1 at the terminal node | 248 | 30380 |
| 15 | $F_4$ | $F_4$ scheme with label 1 at the first node | 26 | 273 |
| 16 | $F_4$ | $F_4$ scheme with label 1 at the last node | 52 | 1274 |
| 17 | $G_2$ | $G_2$ scheme with label 1 at one node | 14 | 77 |
| 18 | $G_2$ | $G_2$ scheme with label 1 at one node | 7 | 14 |
| 19 | $A_1 \times C_m,\ m \geq 2$ | product scheme: $A_1$ node with label 1, and $C_m$ chain with label 1 at the last node | $4m$ | $3(m-1)(2m+1)$ |

1. For the Kronecker product $\varphi(H^+) \times \widetilde{\varphi}(H^+)$, acting in the space $G^+$ of all complex symplectic matrices, there is an invariant irreducible subspace with senior weight of the representation $2\Lambda$.

2. If

$$\frac{2(\Lambda,\alpha)}{(\alpha,\alpha)} \geq 2,$$

where $\alpha$ is a simple root of $H^+$, then in $G^+$ there is, moreover, an invariant irreducible subspace with senior weight of the representation $2\Lambda - 2\alpha$.

3. If

$$\frac{2(\Lambda,\alpha)}{(\alpha,\alpha)} = 1 \quad \text{and} \quad \frac{2(\Lambda,\beta)}{(\beta,\beta)} = 1,$$

where $\alpha, \beta$ are simple roots of $H^+$, then in $G^+$ there is an invariant irreducible subspace with senior weight of the representation $2\Lambda - \alpha - \beta$.

$3^\circ$. By a part of the irreducible representation $\varphi$ of the semisimple algebra $H^+$ we mean a representation of some (other) semisimple algebra $H'^+$, whose scheme is obtained from the scheme $\varphi$ by deleting arbitrary simple roots together with their numerical marks ([1]).

**Lemma 3.** Let $\varphi'$ be a part of an irreducible orthogonal (symplectic) representation $\varphi$ of the algebra $H^+$, which is an orthogonal representation of the algebra $H'^+$. Let $\Lambda'$, $\Lambda$ be the highest weights of $\varphi'$, $\varphi$. Suppose further that the component of the decomposition of $\varphi' \times \varphi'$ of highest weight

$$2\Lambda' - \sum_{\alpha} k_\alpha \alpha \tag{4}$$

(where the summation is over the simple roots of $H'^+$; $k_\alpha$ are nonnegative integers)

**Table 2**

| No. | Type $H^+$ | Scheme $\varphi$ | $N$-dimension of $\varphi$ | Dimension of the homogeneous space |
|---|---|---|---|---|
| 1 | $A_5$ | diagram | 20 | 175 |
| 2 | $A_1$ | diagram | 4 | 7 |
| 3 | $C_3$ | diagram | 14 | 84 |
| 4 | $A_1 \times A_1$ | diagram | 6 | 15 |

| No. | Type $H^+$ | Scheme $\varphi$ | $N$-dimension of $\varphi$ | Dimension of the homogeneous space |
|---|---|---|---|---|
| 5 | $A_1 \times A_1 \times A_1$ | diagram | 8 | 27 |
| 6 | $A_1 \times A_3$ | diagram | 12 | 60 |
| 7 | $A_1 \times B_m$, $m \geq 2$ | diagram | $2(2m+1)$ | $3m(2m+3)$ |
| 8 | $A_1 \times D_m$, $m \geq 4$ | diagram | $4m$ | $3(m+1)(2m-1)$ |

acts in some subspace of the space of skew-symmetric (symmetric) matrices. Then the decomposition of $\varphi \times \widetilde{\varphi}$ contains a component of highest weight

$$2\Lambda - \sum_{\alpha} k_\alpha \alpha$$

(where the summation is over the simple roots of ${H'}^+$; $k_\alpha$ are the same as in (4), acting–

**Table 3**

| Type $H^+$ | Type $G^+$ | Scheme of embedding $H^+ \subset G^+$ | Dimension of the homogeneous space |
|---|---|---|---|
| $A_1$ | $E_6$ | diagram | 11 |
| $A_2$ | $E_6$ | diagram | 35 |
| $A_2$ | $E_7$ | diagram | 125 |
| $C_4$ | $E_6$ | diagram | 52 |
| $G_2$ | $E_6$ | diagram | 64 |

lying in a certain subspace of the space of skew-symmetric (symplectic) matrices.

Some assertions of this lemma are contained in (1).

Lemmas 1–3 make it possible to solve problem A completely when $\varphi$ is irreducible, $H^+$ is semisimple, and $G^+$ is a classical Lie algebra. If, however, $\varphi$ is reducible or $H^+$ is not semisimple, then all solutions of problem A give symmetric spaces.

Let us state the results. All possible nonsymmetric homogeneous Riemannian spaces with irreducible connected rotation group and motion group locally isomorphic to $SO(N)$ or $Sp(N)$ (the group of all real orthogonal matrices with determinant 1, or the group of all unitary symplectic matrices), up to local isomorphism, are described by the tables of highest weights (for embeddings $\varphi(\mathfrak{H}) \subset SO(N)$ or $\varphi(\mathfrak{H}) \subset Sp(N)$; see Tables 1 and 2).

4°. The solution of problem A for the case in which $G^+$ is an exceptional Lie algebra and the subalgebra $H^+$ is nonregular follows directly from the results of (2) (see Table 3).

Among the listed embeddings $\varphi(H^+) \subset G^+$, only the first and the second give rise to nonsymmetric spaces with irreducible rotation group. The remaining ones give symmetric spaces.

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## References

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[3] O. V. Manturov, *DAN*, **141**, No. 4 (1961).

*Note: Figure translations are in progress. See original paper for figures.*