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D. P. ZHELOBENKO

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**Abstract**

**Full Text**

*MATHEMATICS*

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## **DESCRIPTION OF ALL IRREDUCIBLE REPRESENTATIONS OF AN ARBITRARY CONNECTED LIE GROUP**

*(Presented by Academician P. S. Aleksandrov on 11 IV 1961)*

**I.** In the theory of infinite-dimensional representations of Lie groups, the construction of “induced” representations, introduced in the well-known works of I. M. Gelfand and M. A. Naimark, is widely used; this construction is a convenient model of a representation whose operators are determined by an explicit formula in a certain class of functions on a homogeneous manifold. At the same time, in the theory of finite-dimensional representations there still remains a gap: thus, for a semisimple Lie group only the classification of all irreducible representations is known <sup>(2)</sup>, but, as a rule, an effective construction in the form of a model is not known. More precisely, by analogy with the infinite-dimensional case it is easy to write down a formal expression for the representation operators; the difficulty lies in describing the corresponding class of functions in which the representation is realized.

In the present paper a complete solution of the problem is given for an arbitrary connected Lie group (a description of all finite-dimensional irreducible representations); the exposition is carried out for the most meaningful case of a semisimple complex Lie group, and subsequently it is indicated how to reduce to this case the solution of the problem in the general form. Along the way, for the group “as a whole,” É. Cartan’s infinitesimal result is repeated: a description of the system of invariants ( “highest weight” ) characterizing an irreducible representation of a semisimple Lie group up to equivalence; the use of the notion of simple roots <sup>(3)</sup> makes it possible to propose for this purpose a unified method of classification, without recourse to the special structure of one or another simple Lie algebra.

In view of the existence of “Jordan cells,” one may consider that the description of all irreducible representations is only the first step in the study of all possible linear representations; however, if the group under study is semisimple, then for its representations, according to Weyl’s theorem, the property of complete reducibility holds, and knowledge of all irreducible representations in principle makes it possible to describe any linear representation of this group. The results obtained have applications in the “spectral analysis” of linear representations <sup>(5)</sup>, and also in the study of the structure of the Lie group itself.

## II. General scheme of the theory “as a whole” and the main results.

1. For every connected semisimple complex Lie group there is established a “Gauss decomposition” :

$$G = \overline{ZDZ}, \quad Z \cap DZ = D \cap Z = \{e\}$$

(the bar denotes closure), where  $D$  is a connected abelian group (a Cartan subgroup of the group  $G$ ), and the “root” subgroups  $Z$  and  $Z$  are simply connected and nilpotent; the sets  $ZD$  and  $DZ$  are maximal connected (solvable)

\* A similar method of classification “as a whole” was first applied by R. Godement <sup>(4)</sup> for classical Lie groups.

subgroups in  $G$ ; any two decompositions of this type are conjugate with respect to an automorphism of the group  $G$ . The infinitesimal analogue of this theorem is well known from the theory of Killing–Cartan; the idea of the integration consists in using the remarkable property of the **algebraicity** of a simply connected complex Lie group  $\Lambda$  <sup>(6)\*</sup>. For definiteness we shall assume that the Lie algebra of the group  $Z$  is spanned by the root vectors  $e_\alpha$  with positive roots  $\alpha$ .

2. Every irreducible representation of the group  $G$  is induced by some character  $\alpha(\delta)$  of the subgroup  $D$ :

$$T_g f(z) = \alpha(z, g) f(z \cdot g),$$

where  $\alpha(\tilde{z}, \tilde{g})$  is the value of the character  $\alpha$  on the element  $\tilde{\delta}$  in the Gauss decomposition:  $zg = \tilde{\delta}\tilde{z}$  and  $z \cdot g = \tilde{z}$ ; the linear space  $\mathfrak{R}_\alpha(Z)$ , in which  $T_g$  acts, consists of polynomials in the canonical parameters in the group  $Z$  (recall that in the case of a simply connected nilpotent Lie group the canonical mapping of the algebra into the group is a mapping onto the group as a whole).

3. If  $G$  is simply connected, then its Cartan subgroup  $D$  is a direct product:

$$D = D_1 D_2 \dots D_r,$$

where  $D_i$  is a one-parameter subgroup whose directing vector is one of the simple roots of the Lie algebra of the group  $G$ .

4. Normalizing the canonical parameters  $u_i$  in the subgroups  $D_i$  by the condition  $0 \leq \text{Im } u_i < 2\pi$ , we write an arbitrary character of the group  $D$  in the form

$$\alpha(\delta) = e^{l_1 u_1 + l_2 u_2 + \dots + l_r u_r};$$

the character  $\alpha(\delta)$  is “inductive” with respect to the group  $G$  if and only if all the exponents  $l_i$  are nonnegative integers. The corresponding representation  $\mathfrak{D}$

is the “Young product”  $\mathfrak{D}_1^{l_1} \mathfrak{D}_2^{l_2} \dots \mathfrak{D}_r^{l_r}$  of the basic representations  $\mathfrak{D}_i$  of the group  $G$ ; the simply-connectedness condition is easily discarded.

5. Define the infinitesimal left-shift operators on the group  $Z$  as elements of a complex-analytic representation  $x \mapsto \mathfrak{D}(x)$  of the Lie algebra of the group  $Z$ , and denote by  $\mathfrak{D}_i = \mathfrak{D}(e_{\alpha_i})$  the operator generated by the root vector  $e_{\alpha_i}$ , where  $\alpha_i$  is one of the simple (positive) roots. The main result:

The space  $\mathfrak{R}_\alpha(Z)$  consists of all solutions of the system of differential equations:

$$\mathfrak{D}_i^{l_i+1} f(z) = 0, \quad i = 1, 2, \dots, r,$$

in the class of complex-analytic functions on the group  $Z$ .

If one is dealing with real representations of the group  $G$ , then the system of operators  $\mathfrak{D}_i$  is supplemented by the system of complex-conjugate operators  $\overline{\mathfrak{D}}_i$  (the antianalytic representation of the left shift), and the space  $\mathfrak{R}_\alpha(Z)$  is the analogous null-space in the class of “all” functions on the group  $Z$  (meaning: functions for which Lie differentiation is defined, for example, in the class of all generalized functions on  $Z$ ); elements of the space  $\mathfrak{R}_\alpha(Z)$  are automatically polynomials of degree not exceeding a fixed one.

6. As a consequence, for each simple complex algebra  $L$  the class  $o(L)$  of all connected Lie groups having  $L$  as their Lie algebra is described; more precisely, the center of the universal covering group from the class  $o(L)$  is described.

\* An analogous theorem holds for a real group  $G$ ; in this case  $D$  is a direct product of a compact and a vector group.

The result can also be formulated in terms of the Poincaré group: call the square matrix  $\mathfrak{S}$  with (integer) entries

$$\mathfrak{s}_{\alpha\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)},$$

where  $\alpha$  and  $\beta$  run through the system of all simple roots, the **structure matrix** of the semisimple algebra  $L$ . Then the order of the Poincaré group for any semisimple complex group  $G \in o(L)$  does not exceed  $\det \mathfrak{S}$  and attains this value for the “adjoint” group  $G_0 \in o(L)^*$ ; if the group  $G$  is simple, then its Poincaré group is cyclic, with the exception of the adjoint group with algebra  $\text{Lie } D_{2m}$ ,  $m = 1, 2, \dots$  (in the latter case the Poincaré group is always the direct product of two cyclic groups of order two).

We give the values of  $\det \mathfrak{S}$  and the number  $N(c)$  of distinct locally isomorphic groups in the class  $o(L)$  for all simple algebras  $L$ :

Algebra	$A_n$	$B_n$	$C_n$	$D_n$ ( $n \geq 3$ )	$E_6$	$E_7$	$E_8$	$F_4$	$G_2$
$\det \mathfrak{S}$	$n + 1$	2	2	4	3	2	1	1	1
$N(o)c$	$(n + 1)$	2	2	4	3	2	1	1	1

$c(p)$  denotes the number of distinct divisors of the integer  $p$ , including  $p$  and 1; in particular, from this follows the simple connectedness of the special linear groups constructed by É. Cartan in <sup>(1)</sup>.

7. Representations of a semisimple real Lie group are obtained by “analytic continuation.” Finally, for an arbitrary connected Lie group it turns out that every irreducible representation differs only by a simultaneous multiplier from an irreducible representation of the (semisimple) factor group by the radical; hence, in particular, follows a generalization of H. Weyl’s theorem: a representation of a connected Lie group is completely reducible if and only if its restriction to the radical is completely reducible. In the case of a complex group, an irreducible representation receives a transparent geometric characterization by means of the concept of an “extremal direction,” or vector of “highest weight” (see <sup>(5)</sup>, theorem 1).

III. **Example 1.**  $G = SL(3, C)$ . The Gauss decomposition is the usual decomposition of a matrix into triangular factors; we shall assume that  $D$  consists of the diagonal elements of  $G$ , and  $Z'$  and  $Z$ , respectively, of all “lower” and all “upper” triangular matrices with ones on the main diagonal, for example:

$$z = \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix}.$$

The complex-analytic character  $a(\delta)$  can be represented in the form  $\Delta_1^{l_1} \Delta_2^{l_2}$ , where  $\Delta_1 = \delta_1$ ,  $\Delta_2 = \delta_1 \delta_2$  are the minors of the matrix

$$\delta = \begin{pmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix}, \quad \delta_1 \delta_2 \delta_3 = 1;$$

the system  $\{l_1, l_2\}$  determines an irreducible representation of the group  $G$  only in the case when  $l_1$  and  $l_2$  are nonnegative integers; the corresponding space of the induced representation is singled out by the system of equations  $\mathcal{D}_i^{l_i+1} f(z) = 0$ ,  $i = 1, 2$ , where

$$\mathcal{D}_1 = \frac{\partial}{\partial z_{12}} + z_{23} \frac{\partial}{\partial z_{13}}, \quad \mathcal{D}_2 = \frac{\partial}{\partial z_{23}}.$$

\* That is, for the group of all automorphisms of the algebra  $L$ .

and consists of polynomials in the variables  $z_{12}, z_{13}, z_{23}$ . The generalization to matrices of  $n$ -th order is obvious.

**Example 2.**  $G = SO(2\nu+1, C)$ . The matrices from  $Z$  and  $\bar{Z}$  become triangular for a certain choice of basis—when the scalar product is written in the form

$$(x, y) = x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1;$$

the irreducible representations are constructed quite analogously; in particular,

$$a(z, g) = \Delta_1^{r_1}(zg) \dots \Delta_\nu^{r_\nu}(zg),$$

where  $\Delta_i(g)$  denotes the principal minor of the matrix  $g \in G$ , formed from the first  $i$  rows and the first  $i$  columns; however the minor  $\Delta_\nu(zg)$  turns out to be reducible in the class of polynomials in the variable  $z$ :  $\Delta_\nu(z, g) = |S(z, g)|^2$ , and, accordingly, the parameter  $r_\nu$  may be a half-integer (the series of double-valued representations of the group  $G$ ); in the same way, for the group  $G = SO(2\nu, C)$ , spinors of the first and second kind arise.

The finite-dimensionality of the space  $\mathfrak{R}_\lambda(Z)$  follows from a general theorem valid for an arbitrary nilpotent Lie group, and is explained by the fact that the “simple” root vectors  $e_\alpha$  constitute a system of generators in the Lie algebra of the group  $Z$ .

To formulate this theorem, introduce “normal” coordinates in a connected nilpotent group  $G$  by means of the well-known decomposition

$$G = H_1 H_2 \dots H_m$$

into a product of one-parameter connected subgroups  $H_i$ , each of which is isomorphic to either the circle or the line.  $G$  is simply connected if and only if all the  $H_i$  are simply connected; in the latter case one can also use canonical coordinates in the group  $G$ , which are related to the normal ones recursively and polynomially.

Let  $\{e_\alpha\}$  be a system of generators in the Lie algebra of the group  $G$ , and let  $\mathcal{D}_\alpha = \mathcal{D}(e_\alpha)$  be their images in the left regular representation on  $G$ ; then the space  $\mathfrak{R}_\lambda(G)$ , defined as the set of all solutions of the system of differential equations

$$\mathcal{D}_\alpha^{l_\alpha+1} f(g) = 0,$$

with nonnegative integers  $l_\alpha$ , is finite-dimensional and consists of polynomials in the normal parameters in the group  $G$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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