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Abstract

Full Text

MATHEMATICS

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A CRITERION FOR THE EXPANDABILITY OF FUNCTIONS IN A QUASI-POWER SERIES AND QUASI-ANALYTIC CLASSES OF FUNCTIONS

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We consider the quasi-power series

$$\varphi(t) = \sum_{k=0}^{\infty} a_k \omega_k \left(\frac{t}{u}, \gamma\right), \tag{1}$$

where $t \in (0, u]$, $u > 0$,

$$a_0 = \varphi(u), \qquad a_k = \frac{(-1)^k u^{\gamma_k - 1} \varphi_k(u)}{\prod_{\nu=1}^{k} \gamma_\nu},$$

$$\varphi_0(t) = \varphi(t), \qquad \varphi_1(t) = \varphi'(t), \qquad \varphi_{k+1}(t) = \left(\frac{\varphi_k(t)}{t^{\gamma_k - \gamma_{k-1} - 1}}\right)', \qquad k = 1, 2, \dots, \tag{2}$$

$$0 = \gamma_0 < \gamma_1 \leqslant \gamma_2 \leqslant \cdots, \tag{3}$$

$$\omega_k \left(\frac{t}{u}, \gamma\right) = (-1)^k u^{-\gamma_k} \prod_{\nu=1}^{k} \gamma_\nu \int_u^t t_1^{\gamma_1 - 1}\, dt_1 \int_u^{t_1} t_2^{\gamma_2 - \gamma_1 - 1}\, dt_2 \cdots \int_u^{t_{n-1}} t_n^{\gamma_n - \gamma_{n-1} - 1}\, dt_n =$$

$$= \frac{\prod_{\nu=1}^{k} \gamma_\nu}{2\pi i} \int_C \frac{(t/u)^{-\xi}\, d\xi}{\prod_{\nu=0}^{k} (\xi + \gamma_\nu)}, \qquad k = 0, 1, 2, \dots, \tag{4}$$

the simple contour C surrounding neighborhoods of all poles of the integrand in (1).

Definition 1. The lower bound of all numbers μ for which the series

$$t^{\mu}\sum_{k=0}^{\infty}a_k\omega_k\left(\frac{t}{u},\gamma\right)$$

converges uniformly on $[0,u]$ will be called the **order of uniform convergence** of the series (1) on $[0,u]$.

Definition 2. A function $\varphi(t)$ on $(0,u]$, $u>0$, belongs to the **class of functions** $T_{\gamma,\varkappa}$ ($T_{\gamma,\varkappa}(0,u]$), if it is expandable into a quasi-power series (1), convergent on $(0,u]$ to the function $\varphi(t)$, with finite order of uniform convergence $\varkappa$ on $[0,u]$.

Definition 3. An infinitely differentiable function $\varphi(t)$ on $(0,u]$, $u>0$, belongs to the **class of functions** $C_{\gamma,\varkappa}$ ($C_{\gamma,\varkappa}(0,u]$), $\varkappa\geqslant 0$, if, for the given sequence of numbers (3), its

there exists a sequence of functions (2) satisfying the condition

$$\left|\int_x^u \varphi_{n+1}(t)t^{\gamma_n+x_1}\,dt\right|\leq C\prod_{\nu=1}^{n}(x'+\gamma_\nu),\quad n=0,1,2,\dots, \tag{5}$$

where $x\in(0,u]$, x_1 and x' ($x_1>x'>x\geq 0$) are arbitrary numbers, and C is a constant independent of n and x.

Definition 4. The class of functions $C_{\gamma,x}(0,u]$ is called a **quasianalytic class** if, from the vanishing of any function of the class and of all its successive derivatives at any point of the interval $(0,u]$, it follows that $\varphi(t)\equiv 0$ on all of $(0,u]$.

Theorem 1. In order that

$$\varphi(t)\in T_{\gamma,x}(0,u],\quad x\geq 0,$$

it is necessary and sufficient that:

1) $\varphi(t)\in C_{\gamma,x}(0,u]$;
2) the sequence of numbers (3) satisfy the condition

$$\sum_{\nu=1}^{\infty}\frac{1}{\gamma_\nu}=\infty. \tag{3'}$$

Theorem 2. For the quasianalyticity of the class of functions $C_{\gamma,x}(0,u]$, it is necessary and sufficient that the sequence of numbers (3) satisfy condition (3').

From Theorems 1 and 2 it follows:

Theorem 3. In order that

$$\varphi(t) \in T_{\gamma,x}(0,u], \quad u > 0,\ x \geq 0,$$

it is necessary and sufficient that the function $\varphi(t)$ belong to the quasianalytic class of functions $C_{\gamma,x}(0,u]$.

Thus it turns out that the function $\varphi(t)$ is expanded into a series of the form (1), convergent on $(0,u]$ to the function $\varphi(t)$, if and only if it is uniquely determined by the totality of all its successive derivatives at one point of the domain of definition.

We note that Theorem 3 definitively solves Carleman's problem on the representation of functions of quasianalytic classes by an analogue of the Taylor series ([2]).

In the case when $\gamma_\nu = \nu$, $\nu = 0, 1, 2, \ldots$, the series (1) becomes the classical Taylor series of the function $\varphi(t)$ in a neighborhood of the point u.

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REFERENCES

[1] G. V. Badalyan, DAN, **136**, No. 1 (1961). [2] T. Carleman, *Les fonctions quasianalytiques*, Paris, 1926.

Note: Figure translations are in progress. See original paper for figures.