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CYBERNETICS AND CONTROL THEORY

O. B. LUPANOV

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Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

O. B. LUPANOV

ON THE REALIZATION OF FUNCTIONS OF THE ALGEBRA OF LOGIC BY FORMULAS OF BOUNDED DEPTH IN THE BASIS $\&, \vee, \bar{}$

(Presented by Academician M. V. Keldysh on 5 VIII 1960)

An important place in the mathematical problems of cybernetics is occupied by the study of the asymptotic laws of complexity of control systems ⁽¹⁾. The “power” or “productivity” of one or another type of control system may be determined by the complexity of the best realization, by means of these systems, of functions of a certain class. In the present note we consider formulas of bounded depth in the basis $\&, \vee, \bar{}$, including disjunctive and conjunctive normal forms ^(2,3) and their generalizations, and it is shown that, from the point of view of the asymptotics of Shannon functions ^(4,5), formulas of depth 3 are equivalent to formulas of any greater depth (formulas of depth 2, i.e. normal forms, do not possess this property).

Let us define by induction certain special classes of formulas which are generalizations of disjunctive and conjunctive normal forms.

The class $A_{\vee}^0 = A_{\&}^0$ consists of the formulas

$$x_1, x_2, \dots, x_n, \dots, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, \dots$$

The class A_{\vee}^k is defined as follows:

- 1) $A_{\&}^{k-1} \subset A_{\vee}^k$;
- 2) if $F_1 \in A_{\vee}^k, F_2 \in A_{\vee}^k$, then $(F_1 \vee F_2) \in A_{\vee}^k$;
- 3) the class A_{\vee}^k contains no formulas other than those provided for in items 1) and 2).

The class $A_{\&}^k$ is defined in the dual manner ⁽²⁾.

Let further

$$A^{\infty} = \bigcup_{k=0}^{\infty} A_{\vee}^k$$

(obviously,

$$\bigcup_{k=0}^{\infty} A_{\vee}^k = \bigcup_{k=0}^{\infty} A_{\&}^k$$

).

For example,

$$(((x_1 \& x_2) \vee (\bar{x}_1 \& x_3)) \vee x_4) \in A_{\vee}^2;$$

$$((((x_1 \vee (x_2 \& x_3)) \& x_4) \vee \bar{x}_5) \& (\bar{x}_1 \vee \bar{x}_2)) \in A_{\&}^5. \quad (1)$$

We introduce the Shannon functions in the usual way. Let $L_{\xi}^k(f)$ (respectively $L_{\xi}^k(n)$) be the least number of variable symbols sufficient for realizing the function f (respectively, any function of the algebra of logic of n arguments) by a formula from the class A_{ξ}^k ($k = 2, 3, \dots$; $\xi = \vee, \&$

or $k = \infty$ and ξ is the empty symbol). By the principle of duality,

$$L_{\vee}^k(n) = L_{\&}^k(n).$$

We shall denote this common value by $L^k(n)$.

With Shannon's function thus defined (where the number of parentheses does not affect the complexity of a formula), essentially A_{\vee}^2 is the class of disjunctive normal forms, $A_{\&}^2$ is the class of conjunctive normal forms, A_{\vee}^3 is the class of "sums of products of sums" of variables or their negations⁽⁶⁾, etc. In this sense the formulas (1) are "equivalent" to the formulas

$$x_1 x_2 \vee \bar{x}_1 x_3 \vee x_4; \quad ((x_1 \vee x_2 x_3) x_4 \vee \bar{x}_5) (\bar{x}_1 \vee \bar{x}_2).$$

It is easy to see that

$$L^2(n) \geq L^3(n) \geq \dots \geq L^k(n) \geq \dots \geq L^{\infty}(n).$$

Theorem*. If $k \geq 3$,

$$L^k(n) \sim \frac{2^n}{\log_2 n},$$

and for any $\varepsilon > 0$ the fraction of functions f of the arguments x_1, \dots, x_n for which

$$\min(L_{\vee}^k(f), L_{\&}^k(f)) \leq (1 - \varepsilon) \frac{2^n}{\log_2 n},$$

tends to zero as n grows.

This result is, in a certain sense, final, since, as is known, for $n \geq 2$

$$L^2(n) = n \cdot 2^{n-2}.$$

The proof of the theorem, in view of the fact that $L^\infty(f) \leq (1 - \varepsilon)2^n / \log_2 n$, consists in indicating a method of constructing, for each function $f(x_1, \dots, x_n)$ of the algebra of logic, a formula from A_{\vee}^3 for it, containing asymptotically no more than $2^n / \log_2 n$ variable symbols. This method makes it possible, for almost all functions, to construct asymptotically best formulas from A_{\vee}^k ($k \geq 3$), since

$$A_{\vee}^3 \subset A_{\vee}^k.$$

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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REFERENCES CITED

1. S. V. Yablonskii, *Collection: Problems of Cybernetics*, vol. 2, 7 (1959).
2. D. Hilbert, W. Ackermann, *Principles of Theoretical Logic*, IL, 1947.
3. S. V. Yablonskii, *Proceedings of the V. A. Steklov Mathematical Institute, Academy of Sciences of the USSR*, **51**, 5 (1958).
4. J. Riordan, C. E. Shannon, *J. Math. and Phys.*, **21**, No. 2, 83 (1942).
5. C. E. Shannon, *Bell Syst. Techn. J.*, **28**, No. 1, 59 (1949).
6. S. Abhyankar, *Trans. IRE*, EC-7, No. 4, 268 (1958).
7. C. E. Shannon, *Trans. Am. Inst. Electr. Eng.*, **57**, 713 (1938).
8. O. B. Lupanov, *DAN*, **128**, No. 3, 464 (1959).
9. O. B. Lupanov, *Collection: Problems of Cybernetics*, vol. 3, 61 (1960).

* In terms of contact parallel-series circuits, Shannon showed ⁽⁷⁾ that $L^\infty(n) \leq 3 \cdot 2^{n-1} - 2$, and Riordan and Shannon ⁽⁴⁾ showed that for any $\varepsilon > 0$ the fraction of functions f of the arguments x_1, \dots, x_n for which $L^\infty(f) \leq (1 - \varepsilon)2^n / \log_2 n$ tends to zero as n grows, and, consequently, $L^\infty(n) \geq 2^n / \log_2 n$. The author showed ^(8,9) that $L^\infty(n) \sim 2^n / \log_2 n$ and that ⁽⁸⁾ $L^4(n) \sim 2^n / \log_2 n$.

Note: Figure translations are in progress. See original paper for figures.

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