NUCLEAR RIESZ SCALES

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Abstract

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MATHEMATICS

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NUCLEAR RIESZ SCALES

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In the present note we study one special class of nuclear spaces; we call them **centers** (**cocenters**) of **Riesz scales**. We were led to the idea of the naturalness of singling out this class, on the one hand, by the works of S. G. Krein ([1–3]), and, on the other, by the works of M. M. Dragilev ([4–6]).

1. A family of Banach spaces E_α, $-\infty \leqslant a \leqslant \alpha \leqslant A \leqslant \infty$, will be called, following ([2]), a **normal scale** if: 1) for $\alpha \leqslant \beta$, $E_\beta \subset E_\alpha$ and densely in it; 2) for all $x \in E_\beta$, with $\alpha \leqslant \beta$, $\|x\|_\alpha$ is a monotonically increasing continuous logarithmically convex function of the parameter α. We give an example (see ([1]), p. 491, example 1).

Example 1. Let H be a Hilbert space and A a positive operator in H without zero vectors, $\|A\| \leqslant 1$. The family of scalar products

$$(x, y)_\alpha = (A^{-\alpha}x, A^{-\alpha}y)$$

generates a normal scale $\{H_\alpha\}$ of Hilbert spaces H_α, $-\infty < \alpha < \infty$, $H_0 = H$. Such a scale, following ([3]), will be called a **Hilbert scale**.

A pair of normal scales (E_α, F_β) will be called a **Riesz** pair if, for every operator

$$T \in L(E_{\alpha_1}, F_{\beta_1}) \cap L(E_{\alpha_2}, F_{\beta_2}),$$

one can assert that: 1) $T \in L(E_\alpha, F_\beta)$, where

$$\frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} = \frac{\beta - \beta_1}{\beta_2 - \beta_1}, \quad (\alpha - \alpha_1)(\alpha - \alpha_2) < 0;$$

2) $\log \|T\|_{E_\alpha \to F_\beta}$ is a convex function of α*.

Lemma 1. *If the pair (E_α, E_β) is Riesz and the biorthogonal system $\{x'_k, x_k\}$ is an (unconditional) basis in E_a and in E_A simultaneously, then $\{x'_k, x_k\}$ will be an (unconditional) basis also in all E_α, $a \leqslant \alpha \leqslant A$**.*

A normal scale $\{E_\alpha\}$ will be called a **Riesz scale** if the pairs (E_α, E_β), (E_α, H_β), and (H_β, E_α) are Riesz, where $\{H_\beta\}$ is any Hilbert scale. The following propositions follow directly from the definitions.

Proposition 1. *Every Riesz scale of Hilbert spaces $\{H_\alpha\}$ is a Hilbert scale; moreover, if for at least one pair of indices $\alpha_1 < \alpha_2$ the embedding $H_{\alpha_2} \to H_{\alpha_1}$*

is completely continuous, then the operator A generating the scale is completely continuous.

Proposition 2. *If the operator A generating a Hilbert scale is completely continuous, then there exists a common (orthogonal) basis for the spaces H_α of the scale.*

Arguing as in ([7]), we obtain:

Proposition 3. *Let the pair (E_α, H_β) be Riesz and let H_β be a Hilbert scale; if*

$$T \in B(E_{\alpha_1}, H_{\beta_1}) \cap L(E_{\alpha_2}, H_{\beta_2}),$$

then $T \in B(E_\alpha, H_{\beta_\alpha})$ for all α, $(\alpha-\alpha_1)(\alpha-\alpha_2) < 0$. An analogous assertion holds also for the Riesz pair (H_β, E_α).

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* By $L(E,F)$ we denote the space of all continuous linear operators from E into F, and by $B(E,F)$ the space of completely continuous operators.
** For the definitions of unconditional and absolute bases, see ([14]).

2. In what follows we shall be interested in projective (inductive) limits of Riesz scales, i.e., topological linear spaces* of the form $E = \bigcap_{\alpha<\alpha_0} E_\alpha$ $\left(E = \bigcap_{\alpha'_0<\alpha} E_\alpha\right)$, which we shall call **centers** (**cocenters**), finite or infinite according as $\alpha_0 < \infty$ ($\alpha'_0 > -\infty$) or $\alpha_0 = \infty$ ($\alpha'_0 = -\infty$).

Theorem 1. Let $E = \bigcap_{\alpha<\alpha_0} E_\alpha$ be a center of a Riesz scale of Banach spaces with nuclear embeddings**. Then a representation $E = \bigcap_{\beta<\beta_0} H_\beta$ is possible, where $\{H_\beta\}$ is a Hilbert scale and $\beta_0 = 1$, if $\alpha_0 < \infty$; $\beta_0 = \infty$, if $\alpha_0 = \infty$.

Here and below, in the case of an infinite scale we assume that it is an analytic scale with condition C (see ([1]), p. 492).

From Theorem 1 and Proposition 1 it follows that:

Theorem 2. The center of every nuclear Riesz scale $E = \bigcap_{\alpha<\alpha_0} E_\alpha$ has a basis and is isomorphic to the Köthe space $M = M(b_{n\lambda})$, where $b_{n\lambda} = \exp(\lambda\beta_n)$, and in the case of a finite scale $\lambda_0 = 1$ and $\sum e^{-t\beta_n} < \infty$ for every $t > 0$ (or $\beta_n/\log n \to \infty$); while in the case of an infinite scale $\lambda_0 = \infty$ and $\sum e^{-T\beta_n} < \infty$ for some $T > 0$ (or $\lim \beta_n/\log n > 0$).

This realization gives a good apparatus for studying the centers (cocenters) of nuclear Riesz scales.

3. Let us compute the approximation dimensions, i.e., the classes*** $\Psi(E)$ and $\Gamma(E)$, for the centers (cocenters) of nuclear Riesz scales. Recall that the author obtained ([8,9]) the following estimate for the ε-entropy of ellipsoids in Hilbert space:

Lemma 2. Let $\mathcal{E} = \{\xi : \sum |\xi_n|^2 a_n^2 \leq 1\}$, $a_n \uparrow \infty$, be an ellipsoid in l^2, and let $m(t) = \inf\{n : a_n \geq t\}$. Then

$$\int_0^{C/\varepsilon} \frac{m(t)}{t}\, dt \geq H_\varepsilon(\mathcal{E}) \geq \int_0^{1/2\varepsilon} \frac{m(t)}{t}\, dt, \qquad C = 8.$$

From this lemma and Theorem 1 of the note ([15]) it follows that:

Theorem 3. Let $B(t) = \inf\{n : \beta_n \geq t\}$. Then

$$\Psi(E^1) = \bigcup \{\varphi(\varepsilon) : \ \forall T > 0,\ K > 0\ \exists \varepsilon_0 > 0$$

$$\left(\varepsilon < \varepsilon_0 : \ \varphi(\varepsilon) \geq \exp\left(T^{-1} \int_0^{T \log \frac{K}{\varepsilon}} B(t)\, dt \right) \right) \Bigg\},$$

$$\Psi(E^\infty) = \bigcup \{\varphi(\varepsilon) : \ \exists \delta > 0\ \forall K > 0\ \exists \varepsilon_0 > 0$$

$$\left(\varepsilon < \varepsilon_0 : \ \varphi(\varepsilon) \geq \exp\left(\delta^{-1} \int_0^{\delta \log \frac{K}{\varepsilon}} B(t)\, dt \right) \right) \Bigg\}.$$

Here (and below) we denote by E^1 the finite center corresponding to the sequence $\beta_n \uparrow \infty$ ($\beta_n \geq 0$), and by E^∞ the infinite one.

* For the introduction of the topology in these spaces see ([16]).

** For the definitions of nuclear embeddings and nuclear spaces see ([11]).

*** For the definition of the class $\Psi(E)$, introduced by A. N. Kolmogorov ([10]), see ([10,8]). The class $\Gamma(E)$ of numerical sequences, introduced for consideration in ([12]), is defined as follows: $\Gamma(E) = \bigcup \{\{r_n\} : \forall U \exists V\ (r_n d_n(V, U) \to 0);\ U, V$ are neighborhoods of zero in E. For the definition of relative n-widths d_n, see ([8,13]).

Theorem 4. Put $\Gamma_\delta = \{\gamma_n : \gamma_n e^{-\delta \beta_n} \to 0\}$. Then

$$\Gamma(E^1) = \bigcap_{\delta \to 0} \Gamma_\delta,$$

and

$$\Gamma(E^\infty) = \bigcup_{k \to \infty} \Gamma_k.$$

From these propositions it follows:

Theorem 5. The centers of an infinite and a finite Riesz scale cannot be isomorphic.

Using Lemma 2, one can show that the following holds.

Theorem 6. If E and F are Fréchet spaces and $\Gamma(E) = \Gamma(F)$, then $\Psi(E) = \Psi(F)$.

The converse, generally speaking, is false, as the following example shows*.

Example 2. Let E and F be the centers of infinite Riesz scales with sequences $\beta_n = e^{n^2}$ and $\beta'_n = e^{(n+1)^2}$, respectively. Theorems 3 and 4 give: $\Gamma(E) \neq \Gamma(F)$, while $\Psi(E) = \Psi(F)$.

This example shows that even for centers of Riesz scales the class $\Psi(E)$ is not an invariant determining the topology. At the same time the following is true:

Theorem 7. Let E and F be the centers of finite (infinite) nuclear Riesz scales, defined by the sequences β_n and β'_n, respectively. Then the following conditions are equivalent: 1) E and F are isomorphic; 2) $\Gamma(E) = \Gamma(F)$; 3)

$$0 < \underline{\lim}\, \beta_n/\beta'_n \leq \overline{\lim}\, \beta_n/\beta'_n < \infty.$$

Let us note, however, that in the totality of all Fréchet spaces the class Γ is not an invariant determining the topology, as is shown by

Example 3. Let A be the space of functions analytic in the open disk, and let Z be the space of entire functions (i.e., A is the center of a finite, and Z of an infinite, Riesz scale with sequence $\beta_n = n$). It is not hard to compute that $\Gamma(A \times Z) = \Gamma(A)$, whereas the spaces $A \times Z$ and A are not isomorphic —this follows from results of M. M. Dragilev ([6]), of which Theorem 10 is a generalization, and from Theorem 5.

In Examples 2 and 3 (see also ([12])) all spaces have a continuous norm; if this is not required, then one can give simpler examples of this type: let E be the center of a finite (or infinite) Riesz scale with such a sequence β_n that $\overline{\lim}\, \beta_{kn}/\beta_n < \infty$ for every k, for instance $\beta_n = n$, and let F be the direct product of a countable number of spaces E.

4. Above we spoke mainly about centers—projective limits of Riesz scales. Let us note, without repeating ourselves, that all the results of the present note are also valid for cocenters—inductive limits

$$X = \bigcup_{\alpha' < \alpha} X_\alpha, \quad -\infty \leq \alpha'_0,$$

of Riesz scales.

Between centers and cocenters there is a duality relation; more precisely, the following is true:

Theorem 8. Let H_α be a Hilbert scale generated by a completely continuous operator (see Example 1). Then

$$\left(\bigcap_{\alpha<1} H_\alpha\right)' = \bigcup_{-1<\alpha} H_\alpha; \qquad \left(\bigcap_{\alpha<\infty} H_\alpha\right)' = \bigcup_{-\infty<\alpha} H_\alpha,$$

$$\left(\bigcup_{-1<\alpha} H_\alpha\right)' = \bigcap_{\alpha<1} H_\alpha, \qquad \left(\bigcup_{-\infty<\alpha} H_\alpha\right)' = \bigcap_{\alpha<\infty} H_\alpha.$$

Let us note that the property of a space E of being the center (cocenter) of a Riesz scale is, generally speaking, not invariant under passage to subspaces and quotient spaces.

Theorem 9. The space $C^\infty[-1,1]$ (i.e. the center of an infinite Riesz scale with sequence $\beta_n = \log n$) is universal for all

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* This example is given in ([12]) as a space not isomorphic to its maximal subspace.

centers of infinite nuclear Riesz scales. There is no universal center of a finite nuclear Riesz scale.

5. To the centers (and cocenters) of nuclear Riesz scales one can transfer a remarkable result of M. M. Dragilev ([6]) on the canonical form of a basis. In ([6]) it is proved: *in the space A_R of functions analytic in the disk $|z| < R$, every basis $\{f_n\}$ can be so rearranged and renormalized that the operator $T(z^h) = \lambda_n f_{k_n}$ will be continuous and continuously invertible in A_R*; in ([4–6]) a method is indicated for the effective construction of the sequences k_n and λ_n from the basis $\{f_n\}$.

A. Dynin and the author proved ([9]) that *in nuclear Fréchet spaces (and in their strong conjugates) all bases are absolute*. It follows from this that an arbitrary rearrangement and normalization of a basis again lead to a basis. We shall call bases $\{x_n\}$ and $\{f_n\}$ **equivalent** if the operator $Tx_n = f_n$ is continuous and has a continuous inverse in E, and **quasi-equivalent** if they become equivalent after some rearrangement and normalization of one of them. Preserving, on the whole, the structure of M. M. Dragilev' s proof ([4–6]) and overcoming certain additional difficulties, one can prove that the following is true.

Theorem 10 (the theorem on equivalence of bases). *In the center (cocenter) of a nuclear Riesz scale all bases are quasi-equivalent.*

The spaces $A(G)$ of functions analytic in polydisc domains G in the space C^m of m complex variables are also (cf. ([17])) centers of finite Riesz scales; to them corresponds the sequence $\beta_n = n^{1/m}$; the space Z_m of all entire functions in C^m is the center of an infinite Riesz scale with the same sequence $\beta_n = n^{1/m}$. In particular, the theorem on equivalence of bases holds for the spaces $A(G)$ and Z_m.

Whether or not the theorem on equivalence of bases is true for arbitrary nuclear Fréchet spaces is unknown to the author. I note only that the technique connected with approximate dimension, which made it possible to prove ([9]) the theorem on the absoluteness of a basis in the general case, by itself cannot give a complete solution of the question (cf. Example 3).

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Note: Figure translations are in progress. See original paper for figures.