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MATHEMATICS

1961

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Abstract

Full Text

MATHEMATICS

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ABSTRACT AUTOMATA AND PARTITIONING OF FREE SEMIGROUPS

(Presented by Academician P. S. Novikov, 29 IX 1960)

An **abstract automaton** is an object consisting of two sets: the set $A = \{a_\alpha, \alpha \in M\}$, whose elements are called the **internal states** of the automaton, and the set $X = \{x_\beta, \beta \in N\}$, whose elements are called the **input states**, or simply the **inputs**, of the automaton.

For each internal state a_α and each input x_β , a product $a_\alpha x_\beta$ is defined, equal to some internal state a_γ of the automaton ($\gamma = f(\alpha, \beta)$). The automaton is called **finite** if both sets A and X are finite, and **infinite** otherwise.

Construct the free semigroup F with the set of generators x_β ($\beta \in N$), and define the product of the internal states of the automaton by words of this free semigroup. We give the definition by induction: for words of length 1 the corresponding definition is contained in the definition of an abstract automaton; for an arbitrary word $s = s'x_\beta$ of length $n + 1$ ($n \geq 1$) and an arbitrary internal state a_α , we define the product $a_\alpha s$ by the formula $a_\alpha s = (a_\alpha s')x_\beta$. We shall agree to say that the word s takes the automaton from the state a_α to the state $a_\alpha s$. It is useful, moreover, to include among the elements of the free semigroup F the empty word e , and among the internal states of the automaton to single out one state a_0 , which we shall call the **initial state**. The product of any internal state a_α of the automaton by the empty word e is by definition taken to be equal to a_α : $a_\alpha e = a_\alpha$ for all $\alpha \in M$.

For any $\alpha \in M$, denote by F_α the set of all words of the free semigroup F that take the initial state a_0 to the state a_α . It is easy to see that the sets F_α are pairwise disjoint and have the property that, for any input x_β and any set F_α , the set of words $F_\alpha x_\beta$ of the free semigroup F is wholly contained in some set F_γ (the index γ here coincides with the index of the internal state $a_\gamma = a_\alpha x_\beta$).

An automaton for which all the sets F_α are nonempty is naturally called **connected**. In such an automaton, a transition (under the action of a suitable input word) from the initial state to any internal state is possible. It is easy to see that a connected automaton is completely determined by specifying the sets F_α .

In view of what has been said, it is natural to introduce the following definition.

Definition 1. Let F be a free semigroup with identity, with system of generators $x = \{x_\beta, \beta \in N\}$. A partition of the semigroup F into nonempty pairwise disjoint sets F_α ($\alpha \in M$) is called an **automaton partition** if, for every generator x_β and every set F_α , the product $F_\alpha x_\beta$ is contained wholly in one of the sets F_γ of the same partition.

It turns out to be useful also to consider a special case of automaton partitions, namely the so-called **semigroup partitions**.

Definition 2. A partition of the free semigroup F with identity into nonempty pairwise disjoint subsets F_α ($\alpha \in M$) is called **semigroup** if, for any α and β in M , the product $F_\alpha F_\beta$ (regarded as the product of subsets of the semigroup) is wholly contained in one of the sets of the given partition.

It is easy to see that the following theorem is true:

Theorem 1. Let $\{F_\alpha, \alpha \in M\}$ be an arbitrary automaton partition of the free semigroup F with generators x_β ($\beta \in N$). Define an automaton with set of internal states $\{a_\alpha, \alpha \in M\}$ and with set of inputs $\{x_\beta, \beta \in N\}$, setting, for arbitrary $\alpha \in M$ and $\beta \in N$, the product $a_\alpha x_\beta$ equal to the state a_γ , whose index coincides with the index γ of that set F_γ for which the inclusion

$$F_\alpha x_\beta \subseteq F_\gamma$$

holds. If F_0 is the set of our partition containing the empty word, then for any $\alpha \in M$ the set F_α coincides with the set of words that take the automaton from the state a_0 to the state a_α .

Proof. From the definition of an automaton partition, by induction on the length of a word it is easy to infer that, for any word s from F , the set $F_0 s$ is wholly contained in one of the sets F_α of the given partition. Since the empty word e is contained in the set F_0 , we have $s \in F_0 s \subseteq F_\alpha$. It is then clear that the inclusion $F_0 s \subseteq F_\alpha$ will hold for any word s from F_α . Consequently, all words of the set F_α take the automaton from the state a_0 to the state a_α . Since the same is true also for the other states, the set F_α contains all words that take the automaton from the state a_0 to the state a_α . The theorem is proved.

Definition 3. An automaton constructed from an automaton partition by the method indicated in Theorem 1 is called an **automaton corresponding to the given partition**.

It follows from the theorem proved that the problem of studying connected abstract automata is completely equivalent to the problem of studying automaton partitions of free semigroups. Indeed, every connected automaton determines a certain automaton partition, from which, by virtue of the theorem proved, the original automaton is uniquely restored.

A second important consequence following from Theorem 1 is the reduction of the problem of representing events in an abstract automaton in the sense of Kleene-Medvedev^(1,2) to the problem of embedding, into an arbitrary partition of the free semigroup with identity, an automaton partition.

In fact, specifying any event S over the input alphabet $\{x_\beta, \beta \in N\}$ specifies a partition of the free semigroup F with generators x_β ($\beta \in N$) into two disjoint subsets: the set S and its complement \bar{S} in the semigroup F . If $\{F_\alpha, \alpha \in M\}$ is an automaton partition embedded in the partition $\{S, \bar{S}\}$, then, by virtue of Theorem 1, the event S is represented in the automaton corresponding to the partition $\{F_\alpha, \alpha \in M\}$ by all those states a_γ for which the corresponding subsets F_γ of the automaton partition are contained in S .

The considerations developed above make it possible to prove, relatively simply, the uniqueness theorem for the minimal automaton representing given events. For this purpose we introduce the natural notion of the intersection and union of partitions.

Let two partitions be given: $G = \{G_\mu, \mu \in P\}$ and $H = \{H_\nu, \nu \in Q\}$ of some set F into nonempty pairwise disjoint subsets. The **intersection** of these partitions will be the partition of the set F into the collection of all nonempty intersections of the form $G_\mu \cap H_\nu$, ($\mu \in P, \nu \in Q$).

To define the operation of union of the partitions G and H , we introduce the notion of GH -connectedness of subsets of the set F . A subset $K \subset F$ will be

call a subset GH -connected if for any pair of elements a and b of this subset one can construct a finite sequence of elements c_1, c_2, \dots, c_n of this same subset such that two conditions are satisfied: 1) $c_1 = a$; $c_n = b$; 2) for every $i = 1, 2, \dots, n-1$, the elements c_i and c_{i+1} are contained either in one and the same set G_μ of the partition G , or in one and the same set H_ν of the partition H . A sequence satisfying these properties will be called a GH -chain connecting the elements a and b .

Definition 4. The partition of the set F into maximal GH -connected subsets K_τ is called the **union** of the partitions Q and H .

From the definition of GH -connectedness it follows immediately that distinct maximal GH -connected subsets of the set F are pairwise disjoint (two intersecting GH -connected subsets together again give a GH -connected subset).

Theorem 2. *The intersection and the union of any two automaton (semigroup) partitions of the free semigroup F are again an automaton (respectively, semigroup) partition.*

Proof. The case of the intersection of two partitions is obvious. Consider the case of the union Q of two partitions G and H . Let a and b be arbitrary elements of one and the same set of the partition Q ; let c and d be arbitrary elements of the same or of any other set of the partition Q ; and let f be an arbitrary element of F . If $a_0 = a, a_1, \dots, a_n = b$ is a GH -chain connecting the elements a and b , and $c_0 = c, c_1, \dots, c_m = d$ is a GH -chain connecting the elements c and d , then, in the case when the partitions G and H are automaton partitions, the chain $a_0f = af, a_1f, \dots, a_n f = bf$ is, obviously, a GH -chain connecting the elements af and bf . Consequently, the union of automaton

partitions is itself an automaton partition. If, however, G and H are semigroup partitions, then the chain $ac = a_0c_0, a_1c_0, \dots, a_{nc}0, a_{nc}1, a_{nc}2, \dots, a_{nc}m = bd$ is a GH -chain connecting the elements ac and bd . Consequently, the union of semigroup partitions will itself be a semigroup partition. The theorem is proved.

It follows immediately from Theorem 2 that the union of any number of automaton (semigroup) partitions, each of which is inscribed in each partition from some fixed set M of arbitrary partitions of the semigroup, will again be an automaton (respectively, semigroup) partition inscribed in each of the partitions of the set M .

As shown by the author in work ⁽³⁾, any finite set of so-called regular events is representable in a finite automaton. Hence, by virtue of Theorem 2, we obtain the following result:

Theorem 3. *A finite automaton with a minimal number of states representing any finite set of regular events is determined by this set uniquely up to isomorphism.*

If one agrees to call a **semigroup automaton** an automaton corresponding to a semigroup partition of the free semigroup, then, for the representation of regular events in semigroup finite automata, there will obviously hold a theorem analogous to Theorem 3.

Received
11 VII 1960

CITED LITERATURE

¹ S. C. Kleene, in: *Automata*, IL, 1956, pp. 15–67. ² Yu. T. Medvedev, *ibid.*, pp. 385–401. ³ V. M. Glushkov, *Ukr. Math. J.*, **12**, No. 2, 147 (1960).

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