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Abstract

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ON POLYNOMIALS ORTHOGONAL WITH WEIGHT FUNCTIONS VANISHING OR BECOMING INFINITE AT INDIVIDUAL POINTS OF THE INTERVAL OF ORTHOGONALITY

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Among questions in the application of the theory of orthogonal polynomials, an important place is occupied by the problem of convergence of Fourier–Chebyshev series in polynomials orthogonal with arbitrary weight functions. For a general solution of this problem it is necessary to know the asymptotic properties of the polynomials as their indices increase without bound, and the asymptotics must be valid on the entire interval of orthogonality.

The cases of systems (apart from the classical ones) for which such asymptotics have been obtained are few in number (^{1,2}), and in recent years, as noted by Szegő (^{3}), only slight progress has been made in this direction. Among the known analytic devices for obtaining asymptotic representations there are none suitable for systems of polynomials orthogonal with weight functions that vanish or become infinite at individual points inside the interval of orthogonality.

The apparatus of differential-return equations that we propose is promising in this respect.

Theorem. Let the function $\rho(x)$ on the interval $[-1, +1]$ be nonnegative and

1) $\rho(x) = O\left(|x - x_k|^{\gamma_k}\right)$ as $x \to x_k$, where

$$-1 = x_1 < x_2 < \cdots < x_r = +1, \quad \gamma_k > -1, \quad k = 1, \dots, r$$

$(\gamma_1 = \beta,\ \gamma_r = \alpha)$;

2) $\rho(x) \geq \rho_0 > 0$ for all $x_k + \delta \leq x \leq x_{k+1} - \delta,\ k = 1, \dots, r-1,\ \delta > 0$;

3) the functions $\rho(x)a(x)$ and $[\rho(x)a(x)]' - \rho(x)b(x)$ are absolutely continuous and vanish at the endpoints of the interval;

$$\{[\rho(x)a(x)]' - \rho(x)b(x)\}' = P(x)\rho(x),$$

where $P(x)$ is a polynomial of some degree, and

$$a(x) = (x^2 - 1) \prod_{k=2}^{r-1} (x - x_k)^2,$$

$$b(x) = [(\alpha + \beta + 2)x + \alpha - \beta] \prod_{k=2}^{r-1} (x - x_k)^2 +$$

$$+ (x^2 - 1) \sum_{k=2}^{r-1} \gamma_k (x - x_k) \prod_{\substack{i=2 \\ i \neq k}}^{r-1} (x - x_i)^2.$$

If

$$\tilde{y}_n(x) = x^n - S_{nx}^{n-1} + d_{nx}^{n-2} + \cdots, \qquad n = 0, 1, 2, \ldots,$$

are polynomials orthogonal with the weight function $\rho(x)$ on the interval $[-1, +1]$,

then in general they satisfy the differential-recurrence equation

$$a(x)y'' + b(x)y' + c(x,n)y = \sum_{k=1}^{m} A_k \tilde{y}_{n-k}(x), \tag{1}$$

where

$$c(x,n) = -n \left(n + 1 + \alpha + \beta + \sum_{k=2}^{r-1} \gamma_k \right) x^{2r-4} + q_{2r-5} x^{2r-5} + \cdots + q_0,$$

$$m = \max[\deg P(x),\ 2r - 4].$$

Proof. Denote $L[y] = a(x)y'' + b(x)y'$. Clearly,

$$L[\tilde{y}_n]/\tilde{y}_n(x) = -c(x,n) + R_{n-1}(x)/\tilde{y}_n(x),$$

where $R_{n-1}(x)$ is a polynomial of degree not exceeding $n - 1$; therefore

$$R_{n-1}(x) = \sum_{k=1}^{n} A_k \tilde{y}_{n-k}(x).$$

Thus the identity

$$L[\tilde{y}_n] + c(x,n)\tilde{y}_n(x) = \sum_{k=1}^{n} A_k \tilde{y}_{n-k}(x)$$

holds. Multiply this identity by $x^l\rho(x)$, $l = 0, \dots, n-1$, and integrate from -1 to $+1$. Integrating by parts and taking into account the assumptions of the theorem, together with the equalities

$$\int_{-1}^{+1} \rho(x) x^{n+j} \tilde{y}_n(x)\, dx = B_j \int_{-1}^{+1} \rho(x) \tilde{y}_n^2(x)\, dx,$$

we successively find all A_k, $k = 1, \dots, n$; moreover it turns out that $A_k = 0$, $k = m+1, \dots, n$. The theorem is completely proved.

The conditions of the theorem are satisfied, for example, for the weight functions:

1) $\rho(x) = a^{\pi(x)}$, $a > 0$, $\pi(x)$ an arbitrary polynomial;

2)

$$\rho(x) = (1-x)^{\alpha}(1+x)^{\beta} \prod_{k=2}^{r-1} |x - x_k|^{\gamma_k}, \quad -1 < x_k < +1, \quad \alpha, \beta, \gamma_k > -1, \quad k = 2, \dots, r-1.$$

Writing out the differential-recurrence equation in each case causes no difficulty. Thus, for the weight function

$$\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}|x - x_0|^{\gamma}, \quad -1 < x_0 < +1, \quad \alpha, \beta, \gamma > -1,$$

one obtains the equation

$$a(x)y'' + b(x)y' + c(x,n)y = A(n)\tilde{y}_{n-1}(x), \tag{2}$$

where

$$a(x) = (x^2 - 1)(x - x_0)^2,$$

$$b(x) = \{(x^2 - 1)\gamma + [(\alpha + \beta + 2)x + \alpha - \beta](x - x_0)\}(x - x_0),$$

$$c(x,n) = -n(n + 1 + \alpha + \beta + \gamma)(x - x_0)^2 + Q_n(x - x_0) + D_n,$$

$$A(n) = \frac{\alpha_n}{\alpha_{n-1}} \left[(S_{n+1} - S_n)(2n + \alpha + \beta + \gamma) + 2S_{n+1} + \alpha - \beta + \gamma x_0\right],$$

$$\alpha_k = \int_{-1}^{+1} (1-x)^{\alpha}(1+x)^{\beta}|x - x_0|^{\gamma} y_k^2(x)\, dx.$$

If the square bracket on the right-hand side of the equation vanishes for all $n = 0, 1, 2, ...$, then the equation becomes homogeneous, but this proves to be equivalent to the condition $Q_n = 0$, which in turn leads to the expressions

$$S_n = -n\,\frac{\alpha-\beta+\gamma x_0}{2n+\alpha+\beta+\gamma}, \qquad \tilde{y}_n(x_0)\tilde{y}_n'(x_0) = 0, \qquad \gamma \neq 0.$$

Considering the equation in the cases $n = 1, 2$, we see that these conditions are possible only when $x_0 = 0$ and $\alpha = \beta$, i.e., for the case of the weight function

$$\rho(x) = (1-x^2)^{\alpha}|x|^{\gamma},$$

when the equation has the form

$$(x^2-1)x^2y'' + \{(x^2-1)\gamma + 2(\alpha+1)x^2\}xy'$$

$$-\left[n(n+1+2\alpha+\gamma)x^2 + \frac{1-(-1)^n}{2}\gamma\right]y = 0.$$

If $\gamma = 0$, then equation (2) becomes the well-known equation for the Jacobi polynomials.

We shall use equation (2) to obtain an asymptotic representation of polynomials orthogonal with the weight function

$$\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}|x-x_0|^{\gamma},$$

$$-1 \leq x \leq +1, \qquad -1 < x_0 < +1, \qquad \alpha, \beta > -1, \qquad \gamma \geq 0,$$

on the whole interval of orthogonality, according to the scheme developed in our paper (4).

To apply this scheme, we divide the interval $[-1, +1]$ into the parts $[-1, l_1]$, $[l_1, x_0]$, $[x_0, l_2]$, $[l_2, +1]$, each of which contains one singular point of equation (2) at one of its endpoints. In all cases, by means of a suitable change of the independent variable, we shall write equation (2) in one of the following two forms:

$$t^2y'' + tp(t)y' + [\lambda^2t^2r(t) - q(t)]y = R(t),$$

where $t = \pm(x - x_0)$, $t \geq 0$, or

$$ty'' + p(t)y' + [\lambda^2r(t) - q(t)]y = R(t),$$

where $t = 1 \pm x$.

The asymptotic representations for the coefficients $\tilde{y}_n(x)$ entering equation (2) are computed in paper (6), namely:

$$S_n = \frac{\beta - \alpha - \gamma x_0}{2} + O(n^{-1}), \qquad A(n) = O(n^{-1}),$$

$$D_n = \begin{cases} \gamma(x_0^2 - 1)[1 + o(1)], & \tilde{y}_n(x_0) = 0, \\ o(1), & \tilde{y}_n(x_0) \neq 0. \end{cases}$$

The scheme considered in paper (4) is stable with respect to violation of homogeneity of the indicated type for the values $-1 < \alpha, \beta \leq +1$, $0 \leq \gamma < 2$. In this case the integrals additionally arising in the remainder term are convergent, and their order, owing to $A(n)$, is at least n^{-2}. As for the values $\alpha, \beta > +1$, $\gamma \geq 2$, the polynomials obtained in this case are expressed in finite form through polynomials for which $-1 < \alpha, \beta \leq +1$, $0 \leq \gamma < 2$ (see (1), p. 28). Taking all this into account and referring to representation (16) of paper (4), we write out the following asymptotic formulas:

$$\tilde{y}_n(x) \simeq C_1 |x - x_0|^{-\frac{\gamma}{2}} (1 - x)^{-\frac{2\alpha+1}{4}} (1 + x)^{-\frac{2\beta+1}{4}} (1 - x_0)^{\frac{\alpha+1}{2}} (1 + x_0)^{\frac{\beta+1}{2}}$$
$$\times \sqrt{|\arcsin x - \arcsin x_0|}\, J_\nu\Big(\sqrt{n(n + 1 + \alpha + \beta + \gamma)}\, |\arcsin x - \arcsin x_0|\Big), \tag{3}$$

where $-1 < l_1 \leq x \leq x_0$ or $x_0 \leq x \leq l_2 < +1$,

$$\nu = \pm \frac{\sqrt{(\gamma - 1)^2 + 4q(x_0)}}{2}, \qquad q(x_0) = \frac{D_n}{x_0^2 - 1},$$

and the sign is chosen depending on the order of the zero of $\tilde{y}_n(x)$ at the point $x = x_0$, i.e., here one may consider separately the asymptotic representations of polynomials that have, for $x = x_0$, a root for all sufficiently large n ($D_n = \gamma(x_0^2 - 1)[1 + o(1)]$), and of polynomials that do not vanish for $x = x_0$ ($D_n = o(1)$). It is easy to see that $\nu \simeq \frac{\gamma+1}{2}$ and $\nu \simeq \frac{\gamma-1}{2}$, respectively.

Further, for $-1 \leq x \leq l_1 < x_0$ we obtain, putting $\nu = \beta$,

$$\tilde{y}_n(x) \simeq C_2 (1 - x)^{-\frac{2\alpha+1}{4}} (1 + x)^{-\frac{2\beta+1}{4}} (x_0 - x)^{-\frac{\gamma}{2}} (1 + x_0)^{\frac{\gamma}{2}}$$
$$\times \sqrt{\frac{\pi}{2} + \arcsin x}\, J_\nu\left(\sqrt{n(n + 1 + \alpha + \beta + \gamma)}\left(\frac{\pi}{2} + \arcsin x\right)\right), \tag{4}$$

and for $x_0 < l_2 \leq x \leq +1$ we shall have, for $\nu = \alpha$,

$$\tilde{y}_n(x) \simeq C_3(1-x)^{-\frac{2\alpha+1}{4}}(1+x)^{-\frac{2\beta+1}{4}}(x-x_0)^{-\frac{\gamma}{2}}(1-x_0)^{\frac{\gamma}{2}}$$
$$\times \sqrt{\frac{\pi}{2}-\arcsin x}\, J_\nu\left(\sqrt{n(n+1+\alpha+\beta+\gamma)}\left(\frac{\pi}{2}-\arcsin x\right)\right). \qquad (5)$$

If one uses the asymptotic representations of the Bessel functions $J_\nu(z)$ as $z \to 0$ and $z \to \infty$, then one may conclude that the orthonormal polynomials

$$\hat{y}(x) = a_n(x^n - S_{nx}^{n-1} + d_{nx}^{n-2} + \cdots), \qquad \gamma \geq 0,$$

at the point $x = x_0$ have order $n^{\gamma/2}$, and at the endpoints of the interval of orthogonality, respectively, $n^{\beta+1/2}$ and $n^{\alpha+1/2}$ (5,6).

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Note: Figure translations are in progress. See original paper for figures.