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Abstract

Full Text

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ON THE BOUNDEDNESS OF SOLUTIONS OF SOME NONLINEAR DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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In the paper [1] Ezeilo considered the equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + f(x) = p(t) \tag{1}$$

and proved the following assertion. Let $p(t)$ be continuous, and let $f(x)$ be continuously differentiable for all t and x, respectively. Suppose the following conditions are satisfied:

1) $a > 0,\ b > 0$.
2) $f(x)\,\mathrm{sign}\,x > 0$ for $|x| \geq 1$.
3) $\lim_{|x|\to\infty} |f(x)| = \infty$.
4) There exists a constant c $(0 < c < ab)$ such that $f'(x) \leq c$ for $|x| \geq 1$.
5) There exists a constant $A > 0$ such that, for all t,

$$|p(t)| < A, \quad \left| \int^t p(t)\,dt \right| < A.$$

Then one can specify a constant $D > 0$ such that, for any solution of equation (1) with initial data at $t = t_0,\ x = x_0,\ \dot{x} = \dot{x}_0,\ \ddot{x} = \ddot{x}_0$, one has

$$|x| < D, \quad |\dot{x}| < D, \quad |\ddot{x}| < D \quad \text{for } t > T(t_0, x_0, \dot{x}_0, \ddot{x}_0). \tag{2}$$

In the present note we formulate several theorems on the boundedness of solutions of nonlinear equations of the third order. Theorem 1 is a direct generalization of Ezeilo’ s theorem.

1°. Consider the differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + f(x) = P_1(x, \dot{x}, \ddot{x}, t). \tag{3}$$

Put $y = ax + \dot{x},\ z = bx + a\dot{x} + \ddot{x}$; then equation (3) is replaced by the system

$$\dot{x} = y - ax, \quad \dot{y} = z - bx, \quad \dot{z} = -f(x) + P(x, y, z, t). \tag{4}$$

Theorem 1. Let the functions $f(x)$ and $P_1(x, \dot{x}, \ddot{x}, t)$ be continuous and satisfy the uniqueness condition for solutions of equation (3) for all $x, \dot{x}, \ddot{x}, t$. Suppose the following conditions are satisfied:

1) $a > 0$, $b > 0$.
2) $0 < \dfrac{f(x)}{x} < ab$ for $|x| \geq 1$.
3) $\lim_{|x|\to\infty} |f(x) - abx| = \infty$.
4) $\lim_{|x|\to\infty} |f(x)| = \infty$.
5) There exists a constant $A > 0$ such that for all $x, \dot{x}, \ddot{x}, t$ one has $|P_1(x, \dot{x}, \ddot{x}, t)| < A$.

Then one can specify a positive constant D such that, for any solution of equation (3),

$$|x| < D, \qquad |\dot{x}| < D, \qquad |\ddot{x}| < D \quad \text{for } t \geq T(t_0, x_0, \dot{x}_0, \ddot{x}_0), \tag{5}$$

where $t_0, x_0, \dot{x}_0, \ddot{x}_0$ are the initial data of the chosen solution.

The proof of the theorem is based on considering the function

$$v_1 = \frac{1}{2}(a^2 x - ay + z)^2 + \frac{1}{2}(z - bx)^2 + \frac{b}{2}y^2 + a\int_0^x |f(x) - abx|\,dx \tag{6}$$

and its total time derivative, taken by virtue of the differential equations of system (4):

$$\dot{v}_1 = -a(a^2 x - ay + z)^2 + [abx - f(x)][2(a^2 x - ay + z) - bx] + \\ +[(a^2 - b)x - ay + 2z]\,P(x, y, z, t). \tag{7}$$

It can be shown that along every solution, after a sufficiently long interval of time, the function v_1 becomes smaller than some constant quantity independent of the choice of the solution. From this it is already not difficult to derive the assertion of the theorem.

2°. Consider the equation

$$\dddot{z} + a\ddot{z} + \varphi(\dot{z}) + bz = G_1(z, \dot{z}, \ddot{z}, t).$$

We shall assume that $b > 0$. The scale of the quantity t can be changed so that $b = 1$. Therefore we shall study the equation

$$\dddot{z} + a\ddot{z} + \varphi(\dot{z}) + z = G_1(z, \dot{z}, \ddot{z}, t). \tag{8}$$

Put $z = -x,\ y = \dot{x} + ax,\ \varphi(x) = -\psi(-x)$; then we obtain the system

$$\dot{x} = y - ax, \qquad \dot{y} = z - \psi(x) + G(x, y, z, t), \qquad \dot{z} = -x. \tag{9}$$

Theorem 2. Let $\varphi(z)$ and $G_1(z, \dot{z}, \ddot{z}, t)$ be continuous and satisfy the uniqueness condition for solutions of equation (8) for all $z, \dot{z}, \ddot{z}, t$. Suppose, in addition, that the following conditions are fulfilled:

1) $a > 0$.
2) $\dfrac{\varphi(x)}{x} > \dfrac{1}{a}$ for $|x| \geq 1$.
3) $|a\varphi(x) - x| \to \infty$ as $|x| \to \infty$.
4) There exists a constant A such that for all $z, \dot{z}, \ddot{z}, t$ one has

$$|G_1(z, \dot{z}, \ddot{z}, t)| < A.$$

Then there exists $D > 0$ such that, for any solution of equation (8),

$$|z| < D, \qquad |\dot{z}| < D, \qquad \ddot{z} < D \quad \text{for } t \geq T(z_1, \dot{z}_0, \ddot{z}_0, t_0). \tag{10}$$

To prove this theorem one should consider the function (see work ([2]))

$$v_2 = \frac{1}{2}y^2 + \frac{a}{2}z^2 - zx + \int_0^x \psi(x)\,dx \tag{11}$$

and its time derivative, taken by virtue of system (9):

$$\dot{v}_2 = x[x - a\psi(x)] + yG(x, y, z, t). \tag{12}$$

3°. Finally, let us consider the equation

$$\dddot{\xi} + g(\ddot{\xi}) + b\dot{\xi} + a\xi = Q_1(\xi, \dot{\xi}, \ddot{\xi}, t).$$

Assuming $b > 0$, we can change the scale of the time variable t in such a way that in our equation $b = 1$. Therefore, let us consider the equation

$$\dddot{\xi} + g(\ddot{\xi}) + \dot{\xi} + a\xi = Q_1(\xi, \dot{\xi}, \ddot{\xi}, t). \tag{13}$$

Put $x = \ddot{\xi}$, $y = -(\dot{\xi} + a\xi)$, $z = -a\dot{\xi}$; then we obtain the system

$$\dot{x} = y - g(x) + Q(x, y, z, t), \qquad \dot{y} = z - x, \qquad \dot{z} = -ax. \tag{14}$$

Theorem 3. Suppose that the function $g(x)$ is continuously differentiable for all x, and that $Q_1(\xi, \dot{\xi}, \ddot{\xi}, t)$ is continuous and satisfies the uniqueness condition for solutions of equation (13) for all $\xi, \dot{\xi}, \ddot{\xi}, t$. Suppose, moreover, that the following conditions are satisfied:

1) $a > 0$.

2) $g'(x) > a + \varepsilon$ for $|x| \geqslant 1$, where $\varepsilon > 0$ is a constant.

3) There exists a constant $A > 0$ such that

$$|Q_1(\xi, \dot{\xi}, \ddot{\xi}, t)| < A$$

for all $\xi, \dot{\xi}, \ddot{\xi}, t$.

Then one can indicate a constant $D > 0$ such that, for any solution of equation (13),

$$|\xi| < D, \qquad |\dot{\xi}| < D, \qquad |\ddot{\xi}| < D \quad \text{for } t \geqslant T(\xi_0, \dot{\xi}_0, \ddot{\xi}_0, t_0). \tag{15}$$

For the proof one considers the function (see (3))

$$v_3 = \frac{1}{2}(z - x)^2 + \frac{1}{2}y^2 + axy - yg(x) + \frac{1}{2}g^2(x) - a\int_0^x g(x)\,dx. \tag{16}$$

The derivative of this function by virtue of system (14) is equal to

$$\begin{aligned} \dot{v}_3 = -[g'(x) - a][y - g(x)] - (z - x)Q(x, y, z, t) - \\ -[g'(x) - a][y - g(x)]\,Q(x, y, z, t). \end{aligned} \tag{17}$$

4°. From the form of the functions v_1, v_2, v_3 and from the well-known Brouwer theorem on the existence of fixed points, one can derive the assertion of the following theorems.

Theorem 4. Let the conditions of Theorem 1 be satisfied. Let the function $P_1(x, \dot{x}, \ddot{x}, t)$ have period ω in t, i.e.

$$P_1(x, \dot{x}, \ddot{x}, t + \omega) = P_1(x, \dot{x}, \ddot{x}, t).$$

Then equation (3) has at least one ω-periodic solution.

Theorem 5. Let the conditions of Theorem 2 be satisfied and, in addition, let the function G_1 have period ω in t. Then equation (8) has at least one ω-periodic solution.

Theorem 6. Let the conditions of Theorem 3 be satisfied and let the function Q_1 have period ω in t. Then equation (13) has at least one ω-periodic solution.

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CITED LITERATURE

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3. V. A. Pliss, *Some Problems in the Theory of Stability of Motion in the Large*, L., 1958.

Note: Figure translations are in progress. See original paper for figures.