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**Abstract**

**Full Text**

## **Reports of the Academy of Sciences of the USSR**

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**MATHEMATICS**

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### **GEODESIC FLOWS ON COMPACT SURFACES OF NEGATIVE CURVATURE**

*(Presented by Academician A. N. Kolmogorov on 30 VIII 1960)*

In the work of E. Hopf <sup>(6)</sup> the ergodicity of the dynamical system generated by the geodesic flow in the space of line elements of a compact two-dimensional surface of negative curvature was proved. In the present work, by methods analogous to those applied in <sup>(4)</sup>, the spectrum and the character of mixing of this dynamical system are investigated.

§ 1. Let  $\tilde{L}$  be the interior of the unit circle of the plane  $z = x_1 + ix_2$ , endowed with the Riemannian metric

$$ds^2 = \lambda^2(x_1, x_2) \frac{dx_1^2 + dx_2^2}{(1 - x_1^2 - x_2^2)^2}. \quad (1)$$

Assume that: 1) the function  $\lambda(x_1, x_2)$  is differentiable 10 times; 2) there exist positive constants  $a, b$ ,  $a \leq b$ , such that  $a \leq \lambda(x_1, x_2) \leq b$ ; 3) the Gaussian curvature  $K(x_1, x_2)$ , determined by the metric (1), is negative, and there exist positive constants  $c, d$ ,  $c \geq d$ , such that  $-c \leq K(x_1, x_2) \leq -d$ . The geodesic lines of the surface  $\tilde{L}$  are close in their properties to the straight lines of the Lobachevsky plane <sup>(6,7)</sup>. Thus, the notion of a geodesic that is positively asymptotic to a given one and passes through a chosen point has meaning (see, for example, <sup>(6)</sup>).

Let  $\Gamma$  be a discrete subgroup of the group of fractional-linear transformations of the plane  $z$  taking the unit circle onto itself. We shall be interested only in the case when  $\Gamma$  has a compact fundamental domain (in the sense of distances in the Lobachevsky plane). If  $\lambda(x_1, x_2)$  is invariant with respect to all  $\gamma \in \Gamma$ , then the identification of points passing into one another under the action of transformations from  $\Gamma$  leads to a certain compact two-dimensional Riemannian manifold  $L$  of negative curvature.

Consider the spaces  $\widetilde{M}, M$ , formed from the line elements of the surfaces  $\widetilde{L}, L$ , respectively. In  $\widetilde{M}$  and  $M$  one can define the geodesic flow  $\{T^t\}$ , i.e. a one-parameter group of transformations, where the individual transformation  $T^t$  consists in the fact that the line element  $l \in \widetilde{M}$  (or  $l \in M$ ) is shifted along the geodesic determined by it in a given direction by a distance  $t$ . In the spaces  $\widetilde{M}$  and  $M$  there exist measures  $d\tilde{\mu}$  and  $d\mu$ , invariant with respect to the flow  $\{T^t\}$  <sup>(6)</sup>.

§ 2. **Horocycles on the manifolds  $\widetilde{L}, L$ .** By horocycles on  $\widetilde{L}$  we shall mean, by analogy with the Lobachevsky plane, orthogonal trajectories to a family of positively asymptotic geodesics. They are constructed in the following way.\* Let  $l \in \widetilde{M}$ , and let  $z(t)$  be the support of the line element  $T^t l$ .

\* For the first time, as far as we know, horocycles on surfaces of variable negative curvature were constructed in <sup>(7)</sup>. Some properties of horocycles discussed below are also indicated there.

Draw the geodesic circle  $R_t$  with center at  $z(t)$  and radius  $t$ . The boundary of the closure of the aggregate of the points  $R_t$ , as  $t$  runs through all values from 0 to  $\infty$ , will be called the horocycle generated by the line element  $l$ , and will be denoted by  $H(l)$ . If  $G_1$  is a geodesic positively asymptotic to the geodesic  $G(l)$  determined by the line element  $l$ , then the set  $H(l) \cap G_1$  consists of exactly one point. If  $F$  is a subset of  $H(l)$ , then the aggregate of line elements whose base points belong to  $F$ , and whose directions are orthogonal to  $H(l)$  and determine geodesics positively asymptotic to  $G(l)$ , will be called belonging to  $F$ .

Let us note the following properties of horocycles:

1. The horocycle  $H(l')$ , generated by a line element  $l'$  belonging to the horocycle  $H(l)$ , coincides with  $H(l)$ . From this property it follows that a horocycle is an orthogonal trajectory to the family of positively asymptotic geodesics.
2. On any geodesic  $G_1$ , positively asymptotic to  $G(l)$ , the horocycles  $H(l)$  and  $H(T^t l)$  cut out an arc of length  $t$ .
3. The geodesic curvature of a horocycle. To compute the geodesic curvature  $\chi$  of the horocycle  $H(l)$  at the point  $z$ , one must draw through  $z$  the geodesic  $G_1$  positively asymptotic to  $G(l)$ , and consider on  $G_1$  the equation  $\chi' + \chi^2 + K = 0$ . The solution of this equation bounded on the entire geodesic  $G_1$  exists, is unique, and  $\chi(z)$  is the geodesic curvature of the horocycle  $H(l)$  at the point  $z$ . The function  $\chi(z)$  satisfies the inequalities  $\sqrt{d} \leq \chi(z) \leq \sqrt{c}$ .
4. Horocycles make it possible to introduce the following coordinate system in the space  $\widetilde{M}$ . Let  $o$  be an arbitrary point of  $\widetilde{L}$  and let  $l \in \widetilde{M}$ . Then there exists only one geodesic  $G$  passing through  $o$  and positively asymptotic to  $G(l)$  <sup>(6)</sup>. By  $\varphi$  denote the angle which the geodesic  $G$  makes with a fixed direction at  $o$ . Next introduce the quantity  $s$ , equal to the length of the

segment of the geodesic  $G$  between the point  $o$  and the point of intersection of  $G$  with  $H(l)$ , taken with a plus sign if  $o$  lies inside  $H(l)$ , and with a minus sign otherwise, and the quantity  $\rho$ , equal to the length of the arc of the horocycle  $H(l)$  from the base point of  $l$  to the point  $G \cap H(l)$ , taken with a plus sign if the rotation of the geodesic  $G$  about the point  $G \cap H(l)$  to make it coincide with the base of  $l$  occurs counterclockwise, and with a minus sign in the opposite case. The coordinates  $\varphi, s, \rho$  determine a line element of the surface  $\tilde{L}$  uniquely. The invariant measure  $d\tilde{\mu}$  is written in these coordinates in the following way:  $d\tilde{\mu} = f(\varphi, s, \rho) d\varphi ds d\rho$ , where  $f(\varphi, s, \rho)$  is a continuous positive function. Moreover, for any  $d > 0$  there exists a constant  $\alpha(d)$  such that for all line elements whose base points are at a distance from the point  $o$  not exceeding  $d$ ,  $f(\varphi, \rho, s) \geq \alpha(d)$ .

5. It is easy to see that horocycles constructed from two line elements  $l$  and  $l'$ , congruent to one another with respect to the subgroup  $\Gamma$ , are congruent with respect to  $\Gamma$ . Therefore horocycles may be regarded as lines on the surface  $L$ .

§ 3. The investigation carried out below is based on the notion of a Kolmogorov dynamical system or, what is the same thing, a  $K$ -system<sup>(1,2,4)</sup>.

**Theorem 1.** *The geodesic flow in the space  $M$  of line elements of the surface  $L$  is a  $K$ -system.*

**Proof.** Let  $\pi = (l_1, l_2)$  be the set of line elements belonging to the arc  $(z_1, z_2)$  of the horocycle  $H(l_1)$  (or  $H(l_2)$ ),  $z_i$  being the base point of  $l_i$ ,  $i = 1, 2$ . Under the action of the transformation  $T^t$  the set  $\pi$  is transformed into the set  $\pi^t$  of line elements belonging to the arc  $(z_1^t, z_2^t)$  of the horocycle  $T^{tH}(l_1) = H(T^t l)$ .

**Lemma 1.** The limit

$$\lim_{z_2 \rightarrow z_1} \lim_{\tau \rightarrow 0} \frac{1}{\tau} \left( \frac{\rho(z_1, z_2)}{\rho(z_1^\tau, z_2^\tau)} - 1 \right),$$

where  $\rho(z_1^\tau, z_2^\tau)$  is the length of the horocycle segment  $H(T^\tau l)$ , exists and is equal to the geodesic curvature  $\varkappa$  of the horocycle  $H(l_1)$  at the point  $z_1$ .

The result of the lemma is obtained by means of the Gauss-Bonnet formula, applied to the quadrilateral bounded by the horocycle segments  $(z_1, z_2)$  and  $(z_1^\tau, z_2^\tau)$  and by the geodesic arcs joining  $z_1$  with  $z_1^\tau$  and  $z_2$  with  $z_2^\tau$ . From the lemma and inequalities (2) it follows that, for any  $\tau \leq 0$ ,

$$\exp(-\sqrt{d}\tau)\rho(x_1, x_2) \leq \rho(x_1^\tau, x_2^\tau) \leq \exp(-\sqrt{c}\tau)\rho(x_1, x_2). \quad (3)$$

Let now  $R$  be a certain fundamental domain of the subgroup  $\Gamma$ , chosen in the form of a non-Euclidean polygon with a finite number of sides and containing the point  $o$  in its interior (see § 2, item 4). Consider the set  $D_0$  of line elements of the surface  $L$ , whose coordinates satisfy the inequalities

$$0 \leq \varphi_l < 2\pi, \quad -\alpha(1 + \varphi_l) \leq s_l \leq \alpha(1 + \varphi_l), \quad -\beta \leq \rho_{T^{-s_l}l} \leq \beta.$$

By choosing  $\alpha$  and  $\beta$ , one can arrange that, for every  $l \in D_0$  and every nonidentity transformation  $\gamma \in \Gamma$ , one has  $\gamma l \notin D_0$ . Then the set  $D_0$  may be regarded as a subset of  $M$ .

Consider the measurable partition  $\xi$  of the set  $D_0$ , each element of which is specified by fixed values of the coordinates  $\varphi$  and  $s$  and consists of the line elements  $l \in D_0$  for which  $\varphi_l = \varphi$  and  $s_l = s$ . Next, let  $\xi'$  be the measurable partition of the space  $M$ , coinciding with the partition  $\xi$  on the set  $D_0$  and degenerate on the set  $M - D_0$  (i.e.,  $M - D_0$  is an element of the partition  $\xi'$ ). Put  $\xi^0 = \prod_{|t| \leq \tau} T^t \xi'$ ,  $\zeta = \prod_{|t| \leq \tau} T^t \xi'$ , where  $\tau > \beta + 2\pi\alpha$ .

**Lemma 2.** There exists a set  $\Omega \subset M$  of measure zero, invariant with respect to the flow  $\{T^t\}$ , such that two line elements  $l_1 \in M - \Omega$  and  $l_2 \in M - \Omega$  can belong to one element of the partition  $\xi^0$  only if they belong to one and the same horocycle.

The proof of Lemma 2 is simple and is omitted.

**Lemma 3.** The set of elements of the partition  $\xi^0$  consisting of a single point has measure zero. Almost every element of the partition  $\xi^0$  consists of line elements  $l \in M$  belonging to a finite segment of some horocycle.

We indicate the main points of the proof of Lemma 3. Let  $\Xi_k$  be the set of line elements of the partition  $\zeta$  intersecting more than one element of the partition  $T^{-k\tau}\zeta$ . On the basis of (2) and (3), for  $\mu(\Xi_k)$  one can obtain the estimate  $\mu(\Xi_k) \leq C_1 \exp(-C_2 K)$ , where  $C_1$  and  $C_2$  are positive constants. Since  $\sum_k \mu(\Xi_k) < \infty$ , the assertion of the lemma follows from the fact that almost every element of the partition  $\zeta$  belongs to a finite number of the sets  $\Xi_k$ .

**Lemma 4.** For the system of partitions  $\xi^t = T^t \xi^0$  the relation

$$\prod_{-\infty}^{\infty} \xi^t = \varepsilon \pmod{0},$$

holds, where  $\varepsilon$  is the partition of the space  $M$  into individual points.

**Proof.** It is easy to show that there exists a number  $\sigma > 0$  such that the lengths of the horocycle segments making up the elements of the partition  $\xi^0$  do not exceed  $\sigma$ . Then, on the basis of Lemma 1, the lengths of the horocycle segments

cycles that constitute elements of the partition  $\zeta^t$ , are bounded by the constant  $\sigma \exp(-\sqrt{d}t)$ . Therefore, if the linear elements  $l_1$  and  $l_2$  lie, for all  $t$ , in one and the same element of the partition  $\zeta^t$ , then the distance between their carriers is equal to zero and, consequently, they coincide. The assertion of the lemma now follows from V. A. Rokhlin's theorem on bases ((<sup>3</sup>, p. 123)).

Thus, a system of partitions  $\zeta^t, \zeta^t = T^t \zeta^0$ , has been obtained with the following properties: 1)  $\zeta^t \geq \zeta^{t_1} \pmod{0}$  for  $t > t_1$ , which follows from the construction, and 2)

$$\prod_{-\infty}^{\infty} \zeta^t = \varepsilon \pmod{0}$$

—see Lemma 4. To complete the proof of the theorem it remains to show that

$$\bigcap_{-\infty}^{\infty} \zeta^t = \nu \pmod{0},$$

where  $\nu$  is the trivial partition, whose only element is the whole space.

**Corollary.** The dynamical system generated by the geodesic flow  $\{T^t\}$  has countably multiple Lebesgue spectrum and is mixing of all degrees (see (5)).

§ 4. With the aid of the theorem proved above one can transfer to the case under consideration the main result of the work (8).

**Theorem 2.** Let a bounded real function  $f$ , given on  $M$ , satisfy the conditions:

- 1) there exist numbers  $\alpha > 0, \varepsilon > 0, \varepsilon_1 > 0, C_1 > 0, C_2 \geq 0$  such that, for all  $x > 0$ ,

$$\mu \left\{ l : \sup_{l_1: \sigma(l_1, l) < x} \left| \int_0^\alpha f(T^\tau l) d\tau - \int_0^\alpha f(T^\tau l_1) d\tau \right| > \frac{C_1}{\log^{1+\varepsilon} \sigma} \right\} \leq \frac{C_2}{|\log^{1+\varepsilon_1} \sigma|},$$

where  $\sigma(l_1, l)$  is the distance in the Riemannian space  $M$ ;

- 2)

$$D_T(f) = \int_M \left[ \int_0^T f(T^t l) dt \right]^2 d\mu \sim cT, \quad \text{where } c > 0;$$

- 3) for every  $\varepsilon > 0$  there exist  $N(\varepsilon)$  and  $T(\varepsilon)$  such that, for all  $t > T(\varepsilon)$ ,

$$\frac{1}{D_T(f)} \int_{\left\{ l : \left| \int_0^T f(T^\tau l) d\tau \right| > N(\varepsilon) \sqrt{D_T(f)} \right\}} \left[ \int_0^T f(T^\tau l) d\tau \right]^2 d\mu \leq \varepsilon.$$

Then, for any fixed  $s, -\infty < s < \infty$ ,

$$\lim_{T \rightarrow \infty} \mu \left\{ l : \frac{\int_0^T f(T^t l) dt - T\bar{f}}{\sqrt{D_T(f)}} < s \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s \exp\left(-\frac{u^2}{2}\right) du,$$

where

$$\bar{f} = \int_M f(l) d\mu.$$

It can be shown that there exist functions satisfying all the requirements of Theorem 2.

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