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Abstract

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MULTIDIMENSIONAL ELLIPTIC BOUNDARY-VALUE PROBLEMS WITH ONE UNKNOWN FUNCTION

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1. The solvability of the general boundary-value problem for an elliptic equation in a bounded domain of Euclidean space is investigated. A method is indicated for reducing such a problem to a system of integro-differential equations on the boundary of the domain, which makes it possible to apply the results of paper ([1]). The case of a second-order equation is analyzed most fully.

2. **Notation.** G is a bounded domain of Euclidean space R^n $(n > 1)$ with infinitely smooth boundary $\dot{G}$;

$x = (x_1, \dots, x_n) \in R^n$; $D = i^{-1}\left(\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}\right)$; $\alpha = (\alpha_1, \dots, \alpha_n)$ is a set of natural numbers, $|\alpha| = \alpha_1 + \dots + \alpha_n$; $D^\alpha = i^{-|\alpha|}\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$; ξ_x is a tangent vector to the manifold $\dot{G}$ at the point $x \in \dot{G}$; τ_x is the unit vector of the inward normal at the point x; if $\eta = (\eta_1, \dots, \eta_n) \in R^{n*}$, then $\eta^\alpha = \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}$;

$A = \sum_{|\alpha| \le 2k} a_\alpha(x) D^\alpha$ is an elliptic differential polynomial with infinitely differentiable complex coefficients on the set $\bar{G}$, the closure of the set G;

$\sigma_A(\xi_x, z) = \sum_{|\alpha| = 2k} a_\alpha(x) \times (\xi_x + z\tau_x)^\alpha$ is the symbol of the operator A (z is a complex number);

$B_i = \sum_{\beta \le m_i} B_i^{(\beta)} \dfrac{\partial^\beta}{\partial t^\beta}$ $(i = 1, \dots, k)$; $B_i^{(\beta)}$ is a singular operator of order $m_{i\beta} \le m_i - \beta$ on the manifold $\dot{G}$ ([1]);

$\sigma_{B_i}(\xi_x, z) = \sum_{m_{i\beta} + \beta = m_i} \sigma_{B_i^{(\beta)}}(\xi_x) z^\beta$ is the symbol of the operator B_i ($\sigma_{B_i^{(\beta)}}(\xi_x)$ is defined in ([1]));

$E(\bar{G})$ (respectively $E(\dot{G})$) is the Schwartz space of infinitely differentiable functions on the set $\bar{G}$ (respectively $\dot{G}$); $W_2^{(l)}(G)$ (respectively $W_2^{(l-1/2)}(\dot{G})$) (l is any natural number) is the Sobolev space of generalized functions on G (respectively the Slobodetskii space on $\dot{G}$ ([3])).

The system $\mathfrak{A} = \{A, B_1, \dots, B_k\}$ defines the operators

$$\mathfrak{A} : E(\bar{G}) \to E(\bar{G}) \times (E(\dot{G}))^k; \tag{1}$$

$$\mathfrak{A} : W_2^{(l)}(G) \to W_2^{(l-2k)}(G) \times W_2^{(l-m_1-1/2)}(\dot{G}) \times \cdots \times W_2^{(l-m_k-1/2)}(\dot{G}) \quad (2)$$

$$(l \geq \max\{2k, m_1 + 1, \dots, m_k + 1\}).$$

We shall call the operator $\mathfrak{A}$ **elliptic** (cf. ([4])) if for each-

for fixed $\xi_x \neq 0$: a) the roots of the z-polynomial $\sigma_A(\xi_x, z)$ are distributed equally between the upper and lower z-half-planes, and b) the z-polynomials $\sigma_{B_i}(\xi_x, z)$ $(i = 1, \dots, k)$ are linearly independent modulo the z-polynomial

$$\sigma_A^+(\xi_x, z) = \prod_{j \leq k}(z - z_j(\xi_x)),$$

where $z_j(\xi_x)$ $(j = 1, \dots, k)$ are the roots of the z-polynomial $\sigma_A(\xi_x, z)$ lying in the upper z-half-plane. This definition is due to Lopatinskii, who also noted that for $n > 2$ condition a) is always satisfied ([5]).

The following proposition is a modification of the results of ([6]).

Theorem 1. *In order that the operator $\mathfrak{A}$ be elliptic, it is necessary and sufficient that the a priori estimate*

$$\|u\|_l \leq C\left(\|Au\|_{l-2k} + \sum_{i \leq k}\|B_i u\|_{l-m_i-\frac{1}{2}} + \|u\|_0\right), \qquad u \in E(\overline{G}),$$

hold, where $\| \ \|_s$ is the norm in $W_2^{(s)}(G)$; $\| \ \|_{s-\frac{1}{2}}$ is the norm in $W_2^{(s-\frac{1}{2})}(\dot{G})$; C is a constant independent of u.

The following theorem is analogous to Theorem 3 of ([1]) (cf. ([4])).

Theorem 2. *For the ellipticity of the operator $\mathfrak{A}$ it is necessary and sufficient that the following set of conditions hold: a) the generalized solutions of the equation $\mathfrak{A}u = 0$ are infinitely differentiable; b) these solutions form a finite-dimensional subspace; c) the operators (1), (2) are normally solvable; d) the defects of the ranges of these operators are finite and equal.*

Let $\nu_{\mathfrak{A}}$ be the dimension of the space $\mathfrak{A}^{-1}(0)$; let $\rho_{\mathfrak{A}}$ be the defect of the ranges of the operators $\mathfrak{A}$; let $\chi_{\mathfrak{A}} = \nu_{\mathfrak{A}} - \rho_{\mathfrak{A}}$ be the index of the operator $\mathfrak{A}$.

The following proposition is analogous to Theorem 4 of ([1]).

Theorem 3. 1) *The index $\chi_{\mathfrak{A}}$ of an elliptic operator is determined by its symbol*

$$\sigma_{\mathfrak{A}}(\xi_x, z) = \{\sigma_A(\xi_x, z),\ \sigma_{B_1}(\xi_x, z), \dots, \sigma_{B_k}(\xi_x, z)\}.$$

2) *The index $\chi_{\mathfrak{A}}$ is constant under uniformly small changes of the first*

$$2\max\{n, k, m_1, \dots, m_k\}$$

derivatives of the symbol $\sigma_{\mathfrak{A}}(\xi_x, z)$.

3. In this section we indicate a method for transforming the operator $\mathfrak{A}$ into a system $\mathfrak{B}$ of singular operators on the manifold $\dot{G}$.

Consider the remainder $\sigma_i'(\xi_x, z)$ $(i = 1, \dots, k)$ from the division, for fixed $\xi_x \neq 0$, of the z-polynomial $\sigma_{B_i}(\xi_x, z)$ by the z-polynomial $\sigma_A^+(\xi_x, z)$. Let B_i' $(i = 1, \dots, k)$ be a boundary operator with symbol $\sigma_i'(\xi_x, z)$.

Lemma. *The indices of the operators $\mathfrak{A}$ and $\mathfrak{A}' = \{A, B_1', \dots, B_k'\}$ are equal.*

Indeed,

$$\sigma_{B_i}(\xi_x, z) = \sigma_{B_i'}(\xi_x, z) + \sigma_A^+(\xi_x, z) R_i(\xi_x, z).$$

Substituting here, instead of $R_i(\xi_x, z)$, the factor $(1-t)R_i(\xi_x, z)$ $(t \in [0,1])$, we find that $\mathfrak{A}$ and $\mathfrak{A}'$ are connected in the class of elliptic operators, so that it remains to apply Theorem 3.

Now put $v_\beta = \partial^\beta u / \partial \tau^\beta$ $(\beta = 0, 1, \dots, k-1)$. Then the system of operators B_i' $(i = 1, \dots, k)$ is transformed into a system $\mathfrak{B}$ of singular operators acting in the space of vector-functions $(v_0, \dots, v_{k-1})$.

Let

$$\mathfrak{D} = \left\{ A,\ 1,\ \frac{\partial}{\partial \tau}, \dots, \frac{\partial^{k-1}}{\partial \tau^{k-1}} \right\}$$

be the operator corresponding to the first boundary-value problem.

Theorem 4. $\chi_{\mathfrak{A}} = \chi_{\mathfrak{D}} + \chi_{\mathfrak{B}}$.

Let us note that the index of the operator $\mathfrak{D}$ is equal to zero if A is a strongly elliptic operator, and also if A is an operator of second order (for the latter see ([7]), where an outline of the proof is given; however, this follows from the fact that the first boundary-value problem satisfies the ellipticity condition with respect to any elliptic operator A, while the set of elliptic operators of second order is linearly connected ([7]), after which one must use Theorem 3).

The symbol $\sigma_{\mathfrak{A}}(\xi)$ decomposes into the product of an elliptic matrix ([1]) and a nondegenerate diagonal matrix.

Theorem 5. *The elliptic operator $\mathfrak{A} = \{A, B\}$, where A is an operator of second order and the order B is arbitrary, has zero index.*

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Note: Figure translations are in progress. See original paper for figures.