A VARIATIONAL PROBLEM CONNECTED WITH THE MONGE-AMPÈRE EQUATION

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Abstract

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A VARIATIONAL PROBLEM CONNECTED WITH THE MONGE-AMPÈRE EQUATION

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1. As is known ([1]), the Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = \varphi(x, y) \tag{1}$$

can formally be obtained as the Euler equation for certain functionals. Such functionals, for example, are

$$E(u) = \iint_\Omega \left[u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx} + 6\varphi(x, y)u\right]\, dx\, dy; \tag{2}$$

$$I(u) = \iint_\Omega u\left[u_{xx}u_{yy} - u_{xy}^2 - 3\varphi(x, y)\right]\, dx\, dy. \tag{3}$$

Here Ω denotes a domain in the x, y-plane bounded by a closed twice continuously differentiable curve Γ. If the functionals (2), (3) are considered on the class of twice continuously differentiable functions that vanish on Γ, then the identity

$$2I(u) = -E(u),$$

holds, which is obtained by a simple integration by parts in the integral (2).

In the present paper we study questions connected with the solution of the variational problem for the functional (3). As is known, the correct formulation of such problems is connected with requiring that the corresponding Euler equation (1) be elliptic. It is not difficult to see that ellipticity of equation (1) is equivalent to the positivity of the function $\varphi(x, y)$, which in turn entails the convexity of all twice continuously differentiable solutions of equation (1).

Since the Euler equation for the functional (3) has second, and not fourth, order, the variational problem under consideration is degenerate. This fact affects the number of boundary conditions in the variational problem; namely, instead of two boundary conditions one has to consider only one boundary condition. The role of the lost boundary condition is, to a certain extent, replaced by the convexity condition on the functions giving an extremum to the functional $I(u)$. This circumstance plays an essential role in the investigation of the questions that interest us.

2. Let Ω be a convex domain in the x, y-plane bounded by a closed smooth essentially convex curve Γ (i.e., every tangent to Γ has with Γ a single common point). Denote by C_h^+ the collection of continuous nonnegative functions $u(x, y)$ defined in $\Omega+\Gamma$ and assuming on Γ a prescribed continuous nonnegative function $h(X)$. Denote by W_h^+ the class of all convex functions that assume on Γ the values $h(X)$ and are convex toward $z > 0$. Obviously, $W_h^+ \subset C_h^+$. The class W_h^+ is nonempty. Indeed, let Z be the cylinder with directrix Γ and generators parallel to the z-axis. The function $h(X)$ determines on Z a certain closed curve L with one-to-one projection onto Γ. It

divides Z into two cylindrical domains: the lower Z_1 and the upper Z_2. Let us construct the convex hull of Z_1. It is easy to prove that its boundary consists of the cylindrical domain Z_1 and an upward-convex surface of zero exterior curvature S_L. The surface S_L is given by a function $z_L(x, y) \in W_h^+$.

Now let $u \in C_h^+$. Construct the convex hull of Z_1 and of the surface $z = u(x, y)$. The upper boundary of this convex hull is given by some function $\bar{u}(x, y) \in W_h^+$. We shall call the function $\bar{u}(x, y)$ the convex envelope of the function $u(x, y)$. Define the functionals

$$\Phi_1(u) = \iint_\Omega u\omega(\bar{u}, de), \qquad \Phi_2(u) = -3\iint_\Omega \varphi u\, dx\, dy, \qquad I(u) = \Phi_1(u)+\Phi_2(u), \tag{4}$$

where $\omega(\bar{u}, e)$ is the area of the normal image of the convex function $\bar{u}(x, y)$.* The functional $I(u)$ is the extension of the functional (3) to the class of functions $C_h^+(\Omega)$. In what follows we shall consider the functionals $\Phi_2(u)$ and $I(u)$ in the somewhat more general form

$$\Phi_2(u) = -3\iint_\Omega u\mu(de), \qquad I(u) = \Phi_1(u) - 3\iint_\Omega u\mu(de), \tag{5}$$

where $\mu(e)$ is a completely additive nonnegative set function for which $\mu(\Omega) < +\infty$. The functionals (4) are obtained from (5) if $\mu(e)$ is an absolutely continuous set function. Note that if $w \in W_h^+$, then

$$\Phi_1(w) = \iint_\Omega w\omega(w, de), \qquad I(w) = \iint_\Omega w\omega(w, de) - 3\iint_\Omega w\mu(de).$$

Theorem 1. For any function $u \in C_h^+$, the relations

$$\Phi_1(u) = \Phi_1(\bar{u}), \qquad \Phi_2(u) \geq \Phi_2(\bar{u}), \qquad I(u) \geq I(\bar{u}).$$

hold.

It is therefore clear that it is enough to seek a function realizing the absolute minimum of the functional $I(u)$ in the set W_h^+.

Up to this point, in the problem of the minimum of the functional $I(u)$ only one boundary condition, $u|_\Gamma = h$, has been used. It turns out that the functional $I(u)$ is not continuous in the class of functions W_h^+, and still less in C_h^+. The point is that, for the equality $\lim I(u_n) = I(u)$, where $u_n \in W_h^+$ and converge uniformly to $u \in W_h^+$, to hold, it is necessary and sufficient that

$$\lim_{n\to\infty} \omega(u_n, \Omega) = \omega(u, \Omega). \tag{6}$$

Relation (6) is, in a certain sense, the analogue of the discarded second boundary condition in the variational problem under consideration. Namely, condition (6) reduces to the requirement that the supporting planes at boundary points of the convex surfaces $u_n(x, y)$ must converge to the supporting planes on the boundary of the limiting surface $u(x, y)$. In the case of sufficient smoothness of the functions, this means that at points of Γ, $\partial u_n/\partial n \to \partial u/\partial n$ ($\partial u/\partial n$ is the derivative of $u(x, y)$ in the normal direction). Hence it is seen that condition (6) is, in a certain sense, a generalization of the condition that, in a nondegenerate variational problem of second order, the comparison functions are subject to the condition $\partial u/\partial n|_\Gamma = h_1(x)$.

The difficulties connected with the absence of continuity of the functional $I(u)$ are overcome in the following way.

3. Denote by $\mathfrak{M}_\varepsilon$ the totality of all completely additive nonnegative set functions $\mu(e)$ such that $\mu(\Omega) < +\infty$ and $\mu(\Omega_\varepsilon) = 0$, where Ω_ε is the open boundary strip of width $\varepsilon > 0$ of the domain Ω. Further, let $W_{h,\varepsilon}^+$ be the totality of all convex functions in W_h^+ for which $\omega(u, \Omega_\varepsilon) = 0$. Then the following theorems hold:

* $\omega(w, e)$ is defined for any convex function $w(x, y)$ and is a completely additive nonnegative function of Borel subsets of the domain Ω [2].

Theorem 2. For the functional $I(u)$ in the case when $\mu(e) \in \mathfrak{M}_\varepsilon$, it is sufficient to seek a function that gives an absolute minimum to this functional in the class $W_{h,\varepsilon}^+$.

Theorem 3. The class of functions $W_{h,\varepsilon}^+$ is closed with respect to uniform convergence.

Theorem 4. The functional $I(u)$ is continuous on $W^+_{h,\varepsilon}$.

The solution of the problem of finding a function giving an absolute minimum to the functional $I(u)$ is based on the following facts.

Theorem 5. For every function $u \in W^+_{h,\varepsilon}$ for which

$$\|u\|_C \geqslant 2 \max_\Gamma h(X) \geqslant 0,$$

the estimates

$$\Phi_1(u) \geqslant C_1 \varepsilon \|u\|_C \big(\|u\|_C - \max_\Gamma h(x)\big)^2, \qquad |\Phi_2(u)| \leqslant 3\|u\|_C \mu(\Omega),$$

hold, where C_1 is a constant depending only on the domain Ω.

It follows directly from Theorem 5 that $\lim I(u_n) = +\infty$, if $u_n \in W^+_{h,\varepsilon}$ and $\|u_n\| \to +\infty$. Therefore there exists an $M > 0$ such that for all $u \in W^+_{h,\varepsilon}$ for which $\|u\|_C > M$, we have

$$I(u) > 1.$$

On the other hand, for the function $z_L \in W^+_{h,\varepsilon}$ constructed in item 1,

$$I(u) = -3 \iint_\Omega z_L \, \mu(de) \leqslant 0,$$

since $\omega(z_L, \Omega) = 0$. Therefore it is sufficient to seek the function giving the absolute minimum of $I(u)$ in the set $W^+_{h,\varepsilon,M}$ of functions from $W^+_{h,\varepsilon}$ for which $\|u\|_C \leqslant M$. But the set $W^+_{h,\varepsilon,M}$ is compact in the metric $C(\Omega)$, and the functional $I(u)$ is continuous on this set. Therefore there exists a function $u_\varepsilon \in W^+_{h,\varepsilon}$ such that

$$I(u_\varepsilon) = \inf_{u \in C^+_h} I(u).$$

4. Let $u(x, y)$ be some function of the class $W^+_{h,\varepsilon}$. Denote by $\eta(x, y)$ a function twice continuously differentiable in Ω and vanishing in the boundary strip Ω_ε. Further, let α be a real parameter varying in $(-1, 1)$. Consider the functional

$$T(\alpha) = I(u + \alpha\eta).$$

Theorem 6. Under the assumptions of item 4 there exists the derivative

$$\left.\frac{dT}{d\alpha}\right|_{\alpha=0} = \lim_{\alpha\to 0}\frac{I(u+\alpha\eta)-I(u)}{\alpha},$$

and the formula

$$\left.\frac{dT}{d\alpha}\right|_{\alpha=0} = 3\iint_\Omega \eta\,[\omega(u,de)-\mu(de)].$$

holds.

Let u_ε be the function giving an absolute minimum to the functional $I(u)$, for which $\mu(e)\in\mathfrak{M}_\varepsilon$. Then it follows from Theorem 6 that, for every twice continuously differentiable function $\eta(x,y)$ vanishing in the boundary strip Ω_ε, the identity

$$\iint_\Omega \eta\,[\omega(u_\varepsilon,de)-\mu(de)] = 0$$

holds. Hence it follows immediately that the function u satisfies, in the functions of a set, the equation

$$\omega(u_\varepsilon,e)=\mu(e), \tag{7}$$

where e is any Borel set in Ω. If

$$\mu(e)=\iint_e \varphi(x,y)\,dx\,dy,$$

where $\varphi(x,y)$ is a nonnegative summable function, equal to zero in Ω_ε, then, since a convex function has a second differential almost everywhere, almost everywhere in Ω

$$\frac{\partial^2 u_\varepsilon}{\partial x^2}\frac{\partial^2 u_\varepsilon}{\partial y^2}-\left(\frac{\partial^2 u_\varepsilon}{\partial x\partial y}\right)^2=\varphi(x,y). \tag{8}$$

It is proved in [3] that the Dirichlet problem for equation (7) in the class of convex functions W_h^+ has a unique solution. Hence it follows:

Theorem 7. *In the class of functions C_h^+ there exists only one function which gives the absolute minimum of the functional $I(u)$. Here it is assumed that $\mu(e)\in\mathfrak{M}_\varepsilon$. This function belongs to the set $W_{h,\varepsilon}^+$.*

5. Let now $\mu(e)$ be a nonnegative completely additive set function on the domain Ω, with $\mu(\Omega) < +\infty$. Construct the set functions $\mu_\varepsilon(M) = \mu(M \cap \Omega - \Omega_{2\varepsilon})$. Then: 1) $\mu_\varepsilon(M) \in \mathfrak{M}_\varepsilon$; 2) as $\varepsilon \to 0$, μ_ε converge weakly to $\mu(M)$. For each $\mu_\varepsilon(M)$ there exists a solution u_ε of the problem for the absolute minimum of the functional

$$I_\varepsilon(u) = \Phi_1(u) - 3 \iint_\Omega u\mu_\varepsilon(de),$$

which is the solution of the Dirichlet problem for equation (7) with boundary condition $u|_\Gamma = h(X)$. From [(2)] it follows that for $\varepsilon_1 > \varepsilon_2$ everywhere in Ω

$$u_{\varepsilon_1}(x, y) \geq u_{\varepsilon_2}(x, y). \tag{9}$$

Since all u_ε are uniformly bounded in absolute value by a quantity depending only on the number $\mu(\Omega)$, there exists

$$\lim_{\varepsilon \to 0} u_\varepsilon(x, y) = u(x, y).$$

The function $u(x, y)$ tends on the boundary of Ω to $h(X)$ and in Ω satisfies the equation $\omega(u, e) = \mu(e)$. Moreover, using inequality (9), we establish that

$$I(u) = \lim_{\varepsilon \to 0} I(u_\varepsilon).$$

6. All the results of the present work carry over completely to the case of functions of n variables $x_1, x_2, \dots, x_n$, for functionals

$$I(u) = \int_\Omega \cdots \int u \det \left\| \frac{\partial^2 u}{\partial x_i \partial x_k} \right\| dx_1 \cdots dx_n - n \int_\Omega \cdots \int u\varphi \, dx_1 \cdots dx_n. \tag{10}$$

The Euler equation for (10) will be the equation

$$\det \left\| \frac{\partial^2 u}{\partial x_i \partial x_k} \right\| = \varphi(x_1, \dots, x_n).$$

All the results of the work also extend to functionals of the form

$$I(u) = \int_\Omega \cdots \int u \left[\det \left\| \frac{\partial^2 u}{\partial x_i \partial x_k} \right\| - \varphi(x_1, \dots, x_n, u) \right] dx_1 \cdots dx_n,$$

where the function $\varphi(x_1, \dots, x_n, u)$ is absolutely continuous with respect to u_n and, moreover:

1) $\partial\varphi/\partial u \geq 0$; 2) $|\varphi| \leq A + B|u|^{n+1-\varepsilon}$ $(\varepsilon > 0)$, where $A > 0$, $B > 0$ are certain constants. We note that in the case $n > 2$ it is convenient to carry out all considerations in the class of convex functions whose convexity is directed downward, since in this case $\det \|\partial^2 u/\partial x_i \partial x_k\|$ is always non-negative. Correspondingly, one must then consider the minimum of the functional $-I(u)$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.