ON THE STABILITY OF A STRIP UNDER COMPRESSION

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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ON THE STABILITY OF A STRIP UNDER COMPRESSION

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Questions of the stability of a strip under compression were considered in works ([1,2]). It was assumed there that loss of stability is determined mainly by a change in the boundary conditions. This assumption is equivalent to retaining terms of one order of smallness in the boundary conditions and neglecting them in the equilibrium equations. A. Yu. Ishlinskii ([2]) was the first to draw attention to this circumstance.

In the present work the stability of the plane form of equilibrium of a strip is considered on the basis of the equilibrium equations proposed by V. V. Novozhilov ([3]).

Consider a rectangular strip with sides a and b, subjected to a compressive uniform pressure p applied to two opposite sides (Fig. 1).

Fig. 1

Introduce the notation: σ_x, σ_y, τ_{xy} are the stress components; ε_x, ε_y, γ_{xy} are the strain components; u, v are the displacements along the x and y axes, respectively; G is the shear modulus.

We study the case of plane strain, i.e., loss of stability occurs in the plane xOy (Fig. 1). Let us write the relations of the theory of elastic strains of incompressible materials

$$\sigma_x - \sigma_y = 2G(\varepsilon_x - \varepsilon_y), \qquad \tau_{xy} = G\gamma_{xy}. \tag{1}$$

We shall seek the solution of the problem in the form

$$\sigma_x = \sigma_x^0 + \sigma_x', \dots, \quad \varepsilon_x = \varepsilon_x^0 + \varepsilon_x', \dots, \quad u = u^0 + u', \dots . \tag{2}$$

The stressed state before loss of stability is determined by the expressions

$$\sigma_x^0 = -p, \qquad \sigma_y^0 = 0, \qquad \tau_{xy}^0 = 0. \tag{3}$$

For some value of the pressure $p = p^*$, which we shall call critical, several forms of equilibrium are possible. Thus, the solution of the problem reduces to determining the components of the perturbation, to which a prime has been assigned as a superscript.

To solve the problem, one should use the equilibrium equations, the boundary conditions expressing the absence of forces along the lateral sides, and the relation between stresses and strains (1).

The equilibrium equations for the perturbation components in the case of a plane problem are written in the form ([3])

$$\frac{\partial}{\partial x}\left[\sigma_x' - \omega'\tau_{xy}^0\right] + \frac{\partial}{\partial y}\left[\tau_{xy}' - \omega'\sigma_y^0\right] = 0, \tag{4}$$

$$\frac{\partial}{\partial x}\left[\tau_{xy}' + \omega'\sigma_x^0\right] + \frac{\partial}{\partial y}\left[\sigma_y' + \omega'\tau_{xy}^0\right] = 0,$$

where

$$\omega' = \frac{1}{2}\left[\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right].$$

Taking into account expressions (3), we rewrite equations (4):

$$\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}'}{\partial x} + \frac{\partial \sigma_y'}{\partial y} - \frac{p}{2}\frac{\partial}{\partial x}\left[\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right] = 0. \tag{5}$$

Let us write the boundary conditions:

$$\sigma_x \cos\widehat{(nx)} + \tau_{xy}\cos\widehat{(ny)} = 0, \tag{6}$$

$$\tau_{xy}\cos\widehat{(nx)} + \sigma_y\cos\widehat{(ny)} = 0,$$

where $\widehat{(nx)}$ and $\widehat{(ny)}$ are the angles formed by the normal to the side boundary of the strip with the axes x and y. Changes of the rectilinear side boundary of the strip at loss of stability occur due to the perturbation displacements u' and v'. Linearizing expressions (6), we obtain

$$\sigma_y' = 0, \qquad \tau_{xy}' + p\frac{\partial v'}{\partial x} = 0 \tag{7}$$

Fig. 2

Figure 2: Fig. 2

for $y = 0, \quad y = b$.

For the perturbation components, relations (1) are valid by virtue of the linearity of the latter. Setting $u' = \partial\Phi/\partial y$, $v' = -\partial\Phi/\partial x$, we satisfy the incompressibility equation.

Using expressions (5) and (1), we obtain the equation for determining the function $\Phi(x, y)$:

$$(1-\gamma)\frac{\partial^4\Phi}{\partial x^4} + (2-\gamma)\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0, \qquad \left(\gamma = \frac{p^*}{2G}\right). \tag{8}$$

We shall seek the solution of equation (8) in the form

$$\Phi(x, y) = f(y)\cos kx.$$

Fig. 2

To determine the function $f(y)$ we obtain the equation

$$\frac{d^4 f}{dy^4} - (2-\gamma)k^2\frac{d^2 f}{dy^2} + (1-\gamma)k^4 f = 0. \tag{9}$$

Equation (9) is readily integrated. After determining the function $\Phi(x, y)$, we find u' and v'. Using relations (1) and the equilibrium equations (5), we obtain the stress components of the perturbed state. For furth–

the latter it is necessary to have the following expressions:

$$\sigma_y' = \Bigg[Gk^2\Bigg(2C_1\operatorname{ch}ky + 2C_2\operatorname{sh}ky + \frac{2-\gamma}{\sqrt{1-\gamma}}C_3\operatorname{ch}k\sqrt{1-\gamma}\,y + \frac{2-\gamma}{\sqrt{1-\gamma}}C_4\operatorname{sh}k\sqrt{1-\gamma}\,y\Bigg) - \frac{p}{2}\gamma k^2\left(\frac{C_3}{\sqrt{1-\gamma}}\operatorname{ch}k\sqrt{1-\gamma}\,y + \frac{C_4}{\sqrt{1-\gamma}}\operatorname{sh}k\sqrt{1-\gamma}\,y\right)\Bigg]\sin kx, \tag{10}$$

$$\tau_{xy}' = Gk^2\left[2C_1\operatorname{sh}ky + 2C_2\operatorname{ch}ky + (2-\gamma)C_3\operatorname{sh}k\sqrt{1-\gamma}\,y + (2-\gamma)C_4\operatorname{ch}k\sqrt{1-\gamma}\,y\right]\cos kx,$$

$$v' = k\left[C_1 \operatorname{sh} ky + C_2 \operatorname{ch} ky + C_3 \operatorname{sh} k\sqrt{1-\gamma}\, y + C_4 \operatorname{ch} k\sqrt{1-\gamma}\, y\right] \sin kx,$$

where C_i $(i = 1, 2, 3, 4)$ are arbitrary constants.

Substituting relations (10) into the boundary conditions (7), we obtain a homogeneous linear system of algebraic equations with respect to the 4 arbitrary constants. Since, in the case of loss of stability, this system must have a nonzero solution, its determinant must be equal to zero.

Hence we obtain the equation for determining the value of the critical pressure:

$$\frac{\left[2 \operatorname{ch} kb\sqrt{1-\gamma} \cdot \operatorname{ch} kb - 1\right]}{\operatorname{sh} kb\sqrt{1-\gamma} \cdot \operatorname{sh} kb} = \frac{1}{(1+\gamma)\sqrt{1-\gamma}} + (1+\gamma)\sqrt{1-\gamma}. \tag{11}$$

To satisfy the geometric conditions at the edges of the strip, we take $k = m\pi/a$, where m is the number of half-waves $(m \geqslant 1)$.

In Fig. 2 a graph is given of the critical parameter γ_2 as a function of kb, corresponding to equation (11). For comparison, an analogous graph γ_1, corresponding to the characteristic equation of work ([1]), is presented. As can be seen, for small kb $(kb < 0.3)$ the values of the critical parameters γ_1 and γ_2 coincide.

It follows from (10) that the displacement $v(x, y)$ is equal to zero for $x = 0$ and $x = b$; therefore, for small kb, the values of γ can be compared with the Euler critical load for the longitudinal bending of a rod with hinged ends, as was done in work ([2]). These values coincide in the case under consideration.

We give the values of the ratios γ_1/γ_2 for several kb:

kb	0.5	0.7	1	1.2	1.6	2
γ_1/γ_2	1.28	2.2	2.12	2.6	3.0	3.5

Taking account of the compressibility of the material presents no fundamental difficulties, but leads to more complicated calculations.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.