ON SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. V. GRUSHIN

ON SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

(Presented by Academician P. S. Aleksandrov on 20 II 1961)

Let $u(x_1, \dots, x_n)$ be a solution of the partial differential equation with constant coefficients

$$P\left(i\frac{\partial}{\partial x_1}, \dots, i\frac{\partial}{\partial x_n}\right) u(x_1, \dots, x_n) = 0. \tag{1}$$

If $u(x_1, \dots, x_n)$ is an infinitely differentiable function in some domain V, then it sometimes follows from this that $u(x_1, \dots, x_n)$ is an infinitely differentiable function in some larger domain. Thus, M. S. Agranovich proved ([1],[3]) that if $u(x_1, \dots, x_n)$ is infinitely differentiable in some neighborhood of the boundary of the domain V and the domain V is bounded, then $u(x_1, \dots, x_n)$ is infinitely differentiable in the whole domain V.

In the present paper several more cases are indicated in which the infinite differentiability of solutions of equation (1) extends from a smaller domain to a larger one.

Theorem 1. *Let there be a bounded domain V, whose boundary contains a piece of a hyperplane ω of dimension $n-1$. Let Γ be that part of the boundary of the domain V which is not contained in ω, and let Γ_ε be the ε-neighborhood of the set Γ. If some generalized function $u(x_1, \dots, x_n)$ in the domain V is a solution of equation (1) and, for some ε, $u(x_1, \dots, x_n)$ is an ordinary infinitely differentiable function in $\Gamma_\varepsilon \cap V$, then $u(x_1, \dots, x_n)$ is an ordinary infinitely differentiable function throughout the domain V.*

Corollary. Let a generalized function of two variables $u(x_1, x_2)$ be a solution of a differential equation with constant coefficients in some convex domain W. If $u(x_1, x_2)$ is an ordinary infinitely differentiable function in some connected domain $V \subset W$, then $u(x_1, x_2)$ is an ordinary infinitely differentiable function in the convex hull of the domain V.

Theorem 2. *Let V be a bounded domain with boundary Γ. Let W be a closed convex set. If a generalized function $u(x_1, \dots, x_n)$ in the domain $V \setminus W$ is a solution of equation (1) and in some neighborhood of the set $\Gamma \setminus W$, $u(x_1, \dots, x_n)$*

is an ordinary infinitely differentiable function, then $u(x_1, \dots, x_n)$ *is an ordinary infinitely differentiable function throughout the domain* $V \setminus W$.

These two theorems can easily be obtained from the following assertion.

Theorem 3. *Suppose that in* n*-dimensional space there is a hyperplane* $(x, \eta) = 0$, *where* η *is the normal vector. For any integer* $k > 0$ *there exists a fundamental solution* $\mathcal{E}(x)$ *of equation (1) which in the half-space* $(x, \eta) > 0$ *is an ordinary* k*-times continuously differentiable function.*

Proof. Choose the coordinate system so that the vector η has the form $(1, 0, \dots, 0)$. We shall now prove the assertion of the theorem for the case when the hyperplane $x_1 = 0$ is not characteristic for equation (1). The latter means that the polynomial $P(s_1, \dots, s_n)$ corresponding to the differential equation (1) can be written in the form

$$P(s_1, \dots, s_n) = a s_1^p + s_1^{p-1} P(s_2, \dots, s_n) + \dots + P_n(s_2, \dots, s_n),$$

where $a \neq 0$.

The fundamental solution of equation (1) that we need can be constructed by integration by the "Hörmander staircase" method, which is described in detail in paper $^{(3)}$ (see also $^{(2)}$, p. 131). The further arguments differ only slightly from the construction carried out in $^{(3)}$.

Denote by $f_k(\xi)$ the function

$$f_k(\xi) = \begin{cases} (k+n+1)\ln|\xi|, & \text{for } |\xi| \geq 1, \\ 0, & \text{for } |\xi| \leq 1. \end{cases}$$

For fixed $\sigma_2, \dots, \sigma_n$, the equation $P^*(\lambda, \sigma_2, \dots, \sigma_n) = 0$ has p roots $\lambda_1(\sigma_2, \dots, \sigma_n), \dots, \lambda_p(\sigma_2, \dots, \sigma_n)$, which depend continuously on $\sigma_2, \dots, \sigma_n$.

Consider in the complex plane $s_1 = \sigma_1 + i\tau_1$ the curvilinear strip

$$f_k(\sigma_1) + 2l + 2 \geq \tau_1 \geq f_k(\sigma_1) + 2l. \tag{2}$$

For $l = f_k(\sigma_2^2 + \dots + \sigma_n^2) + j$, where $j = 1, 2, \dots, 2p+1$, we obtain $2p+1$ strips. Since the equation $P^*(\lambda, \sigma_2, \dots, \sigma_n) = 0$ has only p roots, in one of these strips, for some $0 < j \leq 2p+1$, the roots will not fall. Denote the l corresponding to this j by $l(\sigma_2, \dots, \sigma_n)$. Since the roots depend continuously on $\sigma_2, \dots, \sigma_n$, $l(\sigma_2, \dots, \sigma_n)$ may be regarded as constant in some neighborhood of the point $(\sigma_2, \dots, \sigma_n)$. We now divide the whole space $(\sigma_2, \dots, \sigma_n)$ into "cubes" so that in each of them $l(\sigma_2, \dots, \sigma_n)$ can be chosen constant. The curve $\tau_1 = f_k(\sigma_1) + 2l(\sigma_2, \dots, \sigma_n) + 1$ in the complex space s_1 will be denoted by $\mathcal{L}(\sigma_2, \dots, \sigma_n)$.

Since the roots of the equation $P^*(\lambda, \sigma_2, \dots, \sigma_n) = 0$ are separated from the contour $\mathcal{L}(\sigma_2, \dots, \sigma_n)$ by at least ρ_k, we have

$$|P^*(s_1, \sigma_2, \dots, \sigma_n)| > \rho_k^p |a|$$

for $s_1 \in \mathcal{L}(\sigma_2, \dots, \sigma_n)$.

As in paper (3), one can now construct a functional on the space Z.

$$(E(\sigma), \psi(\sigma)) = \int \cdots \int \left[\int_{L(\sigma_2,\dots,\sigma_n)} \frac{\psi(s_1, \sigma_2, \dots, \sigma_n)}{P^*(s_1, \sigma_2, \dots, \sigma_n)} \, ds_1 \right] d\sigma_2 \cdots d\sigma_n. \tag{3}$$

Just as in (3), one can show that the Fourier transform of this functional is a fundamental solution of equation (1), which for $x_1 > 0$ is an ordinary function defined by means of the expression

$$\mathcal{E}(x_1, \dots, x_n) = \int \cdots \int \left[\int_{L(\sigma_2,\dots,\sigma_n)} \frac{e^{-x_1 \tau_1} \exp(i \sum x_j \sigma_j)}{P^*(s_1, \sigma_2, \dots, \sigma_n)} \, ds_1 \right] d\sigma_2 \cdots d\sigma_n.$$

But for $x_1 > 0$ this integral is a k-times continuously differentiable function.

In the general case the polynomial $P(s_1, \dots, s_n)$ can be written in the form

$$P(s_1, \dots, s_n) = s_1^q P_0(s_2, \dots, s_n) + s_1^{q-1} P_1(s_2, \dots, s_n) + \cdots + P_q(s_2, \dots, s_n).$$

In the complex space $s_2, \dots, s_n$ we can construct a "Hörmander staircase" H, on which $|P_0(s_2, \dots, s_n)| > C > 0$ (see (3) and (2), p. 131). In this case, for any $\psi(s_1, s_2, \dots, s_n) \in Z$ we shall have

$$\int_H \psi(s_1, s_2, \dots, s_n) \, ds_2 \cdots ds_n = \int \psi(s_1, \sigma_2, \dots, \sigma_n) \, d\sigma_2 \cdots d\sigma_n.$$

Now we can repeat all our arguments, replacing everywhere the points of the space $(\sigma_2, \dots, \sigma_n)$ by points of the manifold H. In particular, instead of (3) we shall have that

$$(E(\sigma), \psi(\sigma)) = \int_H \cdots \int \left[\int_{L(s_2,\dots,s_n)} \frac{\psi(s_1, s_2, \dots, s_n)}{P^*(s_1, s_2, \dots, s_n)} \, ds_1 \right] ds_2 \cdots ds_n.$$

Proof of Theorem 2. Let $\varphi(x)$ be a finite, infinitely differentiable function which is equal to zero outside the domain $V \setminus W$ and, for $x \in V \setminus W$, is equal to one whenever the distance from x to the boundary of the domain $V \setminus W$

is greater than ε. Consider the product $\varphi(x)u(x)$. Since $u(x)$ in $V \setminus W$ is a solution of equation (1), we have

$$P\left(i\frac{\partial}{\partial x_1}, \ldots, i\frac{\partial}{\partial x_n}\right)\varphi(x)u(x) = f_1(x) + f_2(x),$$

where $f_1(x)$ is an infinitely differentiable finite function, and the support of $f_2(x)$ is situated in the ε-neighborhood of the set W. Let $\mathcal{E}(x)$ be an arbitrary fundamental solution of equation (1). In this case

$$\varphi(x)u(x) = (f_1(x) + f_2(x)) * \mathcal{E}(x) = f_1(x) * \mathcal{E}(x) + f_2(x) * \mathcal{E}(x). \qquad (4)$$

The term $f_1(x) * \mathcal{E}(x)$ is an ordinary infinitely differentiable function (see (^{2}), p. 174). Consider

$$f_2(x) * \mathcal{E}(x) = \int_{W+\varepsilon} f_2(\xi)\mathcal{E}(x-\xi)\, d\xi.$$

If $x \notin W + \varepsilon$, then $x - \xi$ lies on one side of some hyperplane ω. Let $\mathcal{E}(x)$ be a fundamental solution which, in the half-space formed by the hyperplane ω, is an ordinary k-times continuously differentiable function. In this case $f_2(x) * \mathcal{E}(x)$ is a $k - q$ times continuously differentiable function, where q is the order of the generalized function $f_2(x)$. Since k can be chosen arbitrarily, $f_2(x) * \mathcal{E}(x)$ is a sufficiently smooth function. From equality (4) we obtain that $u(x)$ is an infinitely differentiable function for $x \in V \setminus W$.

The assertion of Theorem 1 follows from Theorem 2 and the lemma of M. S. Agranovich.

In conclusion, the author expresses gratitude to Prof. G. E. Shilov for posing the problem and for a number of valuable suggestions.

Moscow State University
named after M. V. Lomonosov

Received
14 II 1961

REFERENCES

1. M. S. Agranovich, *DAN*, **128**, No. 3 (1959).
2. I. M. Gelfand, G. E. Shilov, *Spaces of Basic and Generalized Functions*, Moscow, 1958.
3. G. E. Shilov, *UMN*, **14**, issue 5 (89) (1959).