# MATHEMATICS

1961

SovietRxiv

## Abstract

## Full Text

# MATHEMATICS

### B. A. SUBBOTOVSKAYA

# ON THE REALIZATION OF LINEAR FUNCTIONS BY FORMULAS IN THE BASIS $\vee, \&, ^{-}$

*(Presented by Academician A. I. Berg on 20 VIII 1960)*

We shall consider formulas in the basis $\vee, \&, ^{-}$. The number of occurrences of variable symbols in a formula $F$ will be called its **complexity** and denoted by $L(F)$, and the number $\min L(F)$, where the minimum is taken over all formulas $F$ realizing $f$ in the basis $\vee, \&, ^{-}$, by $L(f)$. Obviously, in the language of contact circuits $L(f)$ is the minimum number of contacts sufficient for realizing the function $f$ by series-parallel circuits.

We shall denote the function $\sigma + x_1 + \cdots + x_n \pmod 2$ by the symbol $f_\sigma^n$. As was shown by S. V. Yablonskii ($^1$),

$$L(f_\sigma^n) \leq \frac{9}{8} n^2.$$

In this note it will be shown that

$$L(f_\sigma^n) > c n^{3/2},$$

where $c$ is a certain constant.

For the proof, let us consider a certain special class of formulas in the basis $\&, \vee, ^{-}, 0, 1$. Let $\{x_1, \ldots, x_i, \ldots\}$ be the set of variable symbols. We shall call the expressions $0, 1, \overline{0}, \overline{1}, x_i, \overline{x_i}$ $(i = 1, 2, \ldots)$ $\pi$-formulas and $\sigma$-formulas simultaneously. Let $F_1, \ldots, F_s$ be $\pi$-formulas (respectively, $\sigma$-formulas); then $(F_1 \vee \cdots \vee F_s)$ will be called a $\sigma$-formula (respectively, $(F_1 \& \cdots \& F_s)$ will be called a $\pi$-formula). We denote by $W$ the class of all $\pi$-formulas and $\sigma$-formulas. We shall call the formula $(F_1 \vee \cdots \vee F_s)$ (respectively, $(F_1 \& \cdots \& F_s)$) an **extension** of each of its subformulas $F_i$, which we shall in turn call a **component** of this formula.

Obviously, every component is itself a formula from $W$, and, consequently, every component different from formulas of the form $0, \overline{0}, 1, \overline{1}, x_i, \overline{x_i}$ is itself an extension of its components.

Two formulas $F_1$ and $F_2$ realizing one and the same function will be called **equivalent** (notation $F_1 \sim F_2$). We shall not distinguish two equivalent formulas each of which is obtained from the other by a permutation of the components of some of its subformulas.

Obviously, for every formula $F$ in the basis $\vee, \&, ^{-}$ one can indicate an equivalent formula $F_1$ from $W$ such that $L(F) = L(F_1)$. For example, the formula $(((x_1 \vee x_2)\&\overline{x_3})\&x_4)$ corresponds to the formula $((x_1 \vee x_2)\&\overline{x_3}\&x_4)$ from $W$.

For an arbitrary formula $F$ from $W$, the symbol $L(F)$ will denote, as before, the number of occurrences of variables in $F$.

Let $\widehat{W}$ be the set of formulas from $W$ containing no subformulas of the form $0, 1, \overline{0}, \overline{1}$.

**Lemma 1.** *For every formula $F$, $F \in W$, not equivalent to a constant, one can indicate an equivalent formula $\widehat{F}$ from $\widehat{W}$ such that $L(\widehat{F}) \leq L(F)$.*

The assertion of the lemma follows in a trivial way from the equalities $0 \vee x = x$, $0\&x = 0$, $\bar{0} = 1$, $1 \vee x = 1$, $1\&x = x$, $\bar{1} = 0$.

Let $\psi$ be a subformula of the form $x^{\sigma *}$ of a formula $F$ from $W$. Consider its expansion $(\psi \circ \varphi_1 \circ ... \circ \varphi_s)$ in $F$, where $\circ$ denotes $\vee$ or $\&$. Obviously, for a certain value $\tau$ of the occurrence of the variable $x$ in $\psi$ ($\tau = \sigma$, if $\circ$ is $\vee$, and $\tau = \bar{\sigma}$, if $\circ$ is $\&$), the value of the formula $(\tau^{\sigma} \circ \varphi_1 \circ ... \circ \varphi_s)$ does not depend on the expression $\varphi_1 \circ ... \circ \varphi_s$. We shall call $\tau$ the **determining value** of this occurrence of the variable $x$. A formula in which the expansion of every subformula of the form $x_i^{\sigma}$ $(i = 1, 2, ...)$ has no other occurrences of the variable $x_i$ will be called **normal**.

**Example.** The formula $((x_1 \vee x_2)\&(x_1 \vee 0))$ is normal, while the formula $(((x_1\&x_1) \vee x_2)\&x_3)$ is not normal.

**Remark.** If $x^{\sigma_1}$ and $x^{\sigma_2}$ are two distinct subformulas of a normal formula $F$, then their expansions do not intersect.

Let $f$ be a function of the algebra of logic; denote by the symbol ${}^{\tau}f^{x}$ the function that is obtained from $f$ if in it the variable $x$ is replaced by the constant $\tau$. Obviously, if $f$ does not depend essentially on $x$, then ${}^{\tau}f^{x} = f$. Analogously, if $F$ is a formula, then by the symbol ${}^{\tau}F^{x}$ we shall denote the formula obtained from $F$ by substituting the constant $\tau$ for all occurrences of the variable $x$. If $F$ has no occurrences of the variable $x$, then ${}^{\tau}F^{x}$ coincides with $F$.

**Lemma 2.** *For an arbitrary formula $F$ from $W$, not equivalent to a constant, one can find a normal formula $\tilde{F}$ from $W$ equivalent to it such that*

$$L(\tilde{F}) \leq L(F).$$

**Proof.** Let in the formula $F$ from $W$ there be a subformula $\Phi$ of the form $(\psi \circ \varphi_1 \circ ... \circ \varphi_s)$, where $\psi$ has the form $x^{\sigma}$ and $\tau$ is the determining value of the occurrence of the variable $x$ in $\psi$. Suppose further that the expression $\varphi_1 \circ ... \circ \varphi_s$ contains its occurrences of the variable $x$. Consider the formula $F_1$, obtained from the formula $F$ by replacing in it the subformula $\Phi$ by the formula $\Phi_1$ of the form

$$(\psi \circ {}^{\bar{\tau}}\varphi_1^x \circ \ldots \circ {}^{\bar{\tau}}\varphi_s^x).$$

Obviously, for $F$ and $F_1$ the inequality $L(F_1) < L(F)$ holds. We shall show that $F_1$ and $F$ are equivalent. For this it suffices to show that $\Phi \sim \Phi_1$. But indeed, for $x = \tau$ the values of the formulas $\Phi$ and $\Phi_1$ are determined only by the value of the component ${}^{\tau}\psi^x$ and are equal to $\tau^\sigma$, while for $x = \bar{\tau}$ these values coincide by the construction of the formula $\Phi_1$.

Obviously, by applying the described process, one can satisfy the requirement of the lemma. For example, the formula $(((x_1 \& x_1) \vee x_2) \& x_3)$ is equivalent to the normal formula $(((x_1 \& 1) \vee x_2) \& x_3)$ from $W$.

**Lemma 3.** *If a normal formula $F$ from $\hat{W}$, realizing the function $f$, contains $m$ occurrences of the variable $x$, then there exists a $\tau$ and a formula $F_1$, realizing the function ${}^{\tau}f^x$, such that the inequality*

$$L(F) \geq \frac{3}{2}m + L(F_1). \tag{1}$$

*holds.*

**Proof.** Let $F_1, \ldots, F_m$ be the expansions, respectively, of the subformulas $\psi_1, \ldots, \psi_m$ of the formula $F$ of the form $x^{\sigma_i}$, $1 \leq i \leq m$, and let $\tau_1, \ldots, \tau_m$ be, respectively, the determining values of the occurrences of the variable $x$ in these subformulas. Obviously, no two of these expansions have common occurrences of variables. Let $\tau$ be a constant occurring in the set $\tau_1, \ldots, \tau_m$ at least $m/2$ times. Consider the formula ${}^{\tau}F^x$. Obviously, in it no fewer than $m/2$ subformulas ${}^{\tau}F_i^x$ are equivalent respectively constants $\tau^{\sigma_i}$. Replace in the formula $\tau F^x$ each subformula $\tau F_i^x$ equivalent to the constant $\tau^{\sigma_i}$ by this constant, and denote the resulting formula by $F_1$. Taking into account that in each of the subformulas $F_i$, $i = 1, \ldots, m$, there is at least one occurrence of a variable different from $x$, we conclude that $F_1$ satisfies inequality (1).

**Corollary 1.** Let the variable $x$ have, in a normal formula $F$ from $\hat{W}$ realizing a function $f$ of $n$ variables, a maximal number of occurrences equal to $m$; then there are a $\tau$ and a formula $F_1$ realizing the function $\tau f^x$ for which the inequality

$$L(F) \geqslant \frac{L(F_1)}{1 - 3/2m}.$$

holds.

---

* $x^\sigma$ is $x$ for $\sigma = 1$ and $\bar{x}$ for $\sigma = 0$.

Obviously, the number $m$ considered in the hypothesis of the corollary satisfies the inequality $m \geqslant L(F)/n$. Substituting it in (1), we obtain the required inequality.

Now consider a formula realizing the function $f_\sigma^n$. Obviously, the formula $\tau F_n^x$ realizes some function $f_{\sigma_1}^{n-1}$.

**Corollary 2.** If a formula $\hat{F}_n$ from $\hat{W}$ realizes the function $f_\sigma^n$, then

$$L(\hat{F}_n) \geqslant \prod_{i=2}^{n} \frac{1}{1-3/2i} L(f_\sigma^1). \tag{2}$$

The inequality is proved by induction using Corollary 1 of Lemma 3 and Lemmas 1 and 2.

**Theorem.** If $F_n$ is an arbitrary formula in the basis $\vee$, &, $^-$, realizing the function $f_\sigma^n$, then

$$L(F_n) \geqslant Cn^{3/2}.$$

**Proof.** Obviously, $L(F_n)$ satisfies inequality (2). We have

$$\prod_{i=2}^{n} \frac{1}{1-3/2i} = \exp\left[-\sum_{i=2}^{n} \ln\left(1-\frac{3}{2i}\right)\right] = \exp\left[\sum_{k=2}^{n} \frac{3}{2k} + O(1)\right] \geqslant C_1 n^{3/2},$$

whence

$$L(F_n) \geqslant Cn^{3/2}.$$

Moscow State University
named after M. V. Lomonosov

Received
5 VIII 1960

## REFERENCES

1. S. V. Yablonskii, DAN, **94**, No. 5, 805 (1954).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*