Physics

1961

SovietRxiv

Abstract

Full Text

Physics

O. I. Zavyalov and V. P. Pavlov

Matrix Element of the Reaction $\pi + N \to \pi + \pi + N$ at Low Energies

(Presented by Academician N. N. Bogolyubov, 28 II 1961)

Mandelstam integral representations, together with the unitarity condition, lead to a closed system of equations for the scattering amplitudes and apparently give objective information about the corresponding matrix elements. It is therefore natural to bring in definite ideas about the localization and nature of the singularities of the S-matrix for the description of reactions involving 5 particles. Such investigations have a twofold aim. On the one hand, they may lead to satisfactory expressions for the matrix elements. On the other hand, good results will serve as an argument in favor of the initial assumptions about the analytic behavior of the amplitudes of these processes. In the present paper we shall show that taking into account the nearest singularities of the S-matrix leads to formulas already well tested, obtained from other considerations, and indicates prospects for improving these formulas. We shall concentrate on the production of an additional meson in meson–nucleon collisions. However, by the same method it is easy to consider photoproduction of two mesons on a nucleon and decay-type processes, for example τ- and τ'-decay.

1. **Kinematics.** Let us consider three reactions, isotopically related to one another, of additional-meson production in collisions of a π^--meson and a proton:

$$\pi^- + \mathrm{p} \to \pi^+ + \pi^- + \mathrm{n}; \tag{1}$$

$$\pi^- + \mathrm{p} \to \pi^0 + \pi^0 + \mathrm{n}; \tag{2}$$

$$\pi^- + \mathrm{p} \to \pi^- + \pi^0 + \mathrm{p}. \tag{3}$$

To simplify the kinematic derivations we shall regard the nucleons as scalar particles. We shall dwell in detail on reaction (1); processes (2) and (3) can be treated in a completely analogous way.

Let p_1 be the momentum of the π^+-meson; p_2, the momentum of the final π^--meson; p_3, the momentum of the neutron; $-p_4$, the momentum of the initial

Fig. 1

Figure 1: Fig. 1

π^--meson; $-p_5$, the momentum of the proton. In the usual way we introduce the invariant amplitude A_1 of reaction (1), writing symbolically

$$\hat{S} = \hat{1} + iA_1\delta\Big(\sum p\Big)/2\pi(\omega_1\omega_2\omega_3\omega_4\omega_5)^{1/2}.$$

The amplitude A_1 depends only on invariant combinations of the momenta p_i. From these 5 momenta one can form 15 invariants $u_{ik} = u_{ki} = (p_i + p_k)^2$. The diagonal elements of the matrix u are simply related to the masses of the particles participating in the process: $u_{ii} = 4m_i^2$. On the remaining 10 invariants the law of conservation of momentum imposes 5 relations:

$$\sum_{i\neq k} u_{ik} = m_k^2 + \sum_l m_l^2 \quad (k = 1, 2, 3, 4, 5).$$

Thus, of the 10 variables u_{ik}, only 5 turn out to be independent. As independent variables we choose the following: $s_1 = u_{23}$—the energy in the center-of-mass system of the neutron and the π^--meson; $s_3 = u_{12}$—the energy in the system

the center of mass of the π^+- and π^--mesons; $s_2 = u_{13}$ is the energy in the center-of-mass system of the neutron and the π^+-meson, and two momentum transfers, for example, u_{35} is the momentum transfer between the nucleons; u_{14} is the momentum transfer between the mesons.

Let us also introduce a new notation for the energy in the principal channel of the reaction: $u_{45} = W$. In what follows all variables, except s_1, s_2, s_3, and W, will be called momentum transfers. Let us give the values of the variables s_1, s_2, and s_3 at the threshold energy $W = (M+2\mu)^2$: $s_1^0 = s_2^0 = (M+\mu)^2, \quad s_3^0 = 4\mu^2$; here M is the nucleon mass, μ the meson mass.

An important circumstance is that, near the energy threshold of the reaction, the momentum transfers u_{ik} are substantially smaller than the smallest squared mass of a possible intermediate state corresponding to particles i and k, whereas s_1, s_2, and s_3 are rather large. In what follows this will give us grounds to neglect the dependence of the amplitude A_1 on the momentum transfers, restricting ourselves to studying the behavior of the function A_1 in the variables s_1, s_2, and s_3 at fixed energy W. At energies above threshold $W > (M + 2\mu)^2$, the region of physical values s_1, s_2, s_3 is, in any case, bounded by the conditions $s_1 \geq (M + \mu)^2, \quad s_2 \geq (M + \mu)^2, \quad s_3 \geq 4\mu^2$ and, in view of the relation $s_1 + s_2 + s_3 = W + M^2 + 2\mu^2$, lies inside the shaded triangle of Fig. 1. At threshold energy the triangle contracts to a point.

Fig. 1

II. **Analytic properties and the unitarity condition.** Our aim is to obtain the linear terms of the expansion of the matrix element in the relative momenta of the particles participating in the reaction. Suppose that the amplitude A_1 near threshold is well approximated by the expression

$$A_1(s_1 s_2 s_3) = A_1(s_1^0 s_2^0 s_3^0) + \frac{s_1 - s_1^0}{\pi}\int_{(M+\mu)^2}^{\infty} \frac{\sigma_1(s')\,ds'}{(s'-s_1^0)(s'-s_1-i\varepsilon)} + \frac{s_2 - s_2^0}{\pi}\int_{(M+\mu)^2}^{\infty} \frac{\sigma_2(s')\,ds'}{(s'-s_2^0)(s'-s_2-i\varepsilon)} + \frac{s_3 - s_3^0}{\pi}\int_{4\mu^2}^{\infty} \frac{\sigma_3(s')\,ds'}{(s'-s_3^0)(s'-s_3-i\varepsilon)}, \tag{4}$$

where σ_1, σ_2, and σ_3 are determined from the unitarity condition.

In representation (4) the nearest singularities of the amplitude are taken into account. It is obtained from the "normal" Mandelstam representation for the function A_1 in the variables s_1, s_2, and s_3 at fixed W. In this sense (4) is an analogue of the Chew and Fubini approximation for scattering processes.

An investigation of the graphs of perturbation theory shows that the "normal" Mandelstam representation is not satisfied for diagrams with 5 external lines. Moreover, expression (4) does not take into account pole terms of the form $(s_i - s_i^0)/(s_i - M^2)(s_i^0 - M^2)$ and the dependence of A_1 on the momentum transfers u_{ik}. However, all this is inessential for our approximation. Indeed, the expansion of the amplitude in powers of the momenta is, in fact, an expansion in powers of the expressions $(s_i - s_i^0)^{1/2}$, where the point s_i^0 lies in the physical region. But the corrections to the "normal" Mandelstam representation are of order $(s_i - s_i^0)$ and, consequently, of second order in the momenta, which we do not take into account. The physical values of the momentum transfers u_{ik} lie below the smallest squared mass of the intermediate state corresponding to the particles i and k. We are therefore entitled to expect that the function A_1 is analytic in

variables u_{ik} in the region of their physical values. It follows that in this region it can be expanded in a series in powers of $(u_{ik} - u_{ik}^0)$ and has the form $C_1 + C_2(u_{ik} - u_{ik}^0) \sim C_1 + C_2 k^2$. The constant C_1 is taken into account by the first term on the right-hand side of (4), while the remainder of the series contains powers of the momenta higher than the first. For the same reason we do not take pole terms into account. As the subtraction point in (4) we choose the point of threshold values s_i: $s_1^0 = s_2^0 = (M+\mu)^2$, $s_3^0 = 4\mu^2$. Therefore $A_1(s_1^0 s_2^0 s_3^0)$ is the amplitude of reaction (1) at the threshold energy $W = (M+2\mu)^2$. The imaginary additions in the denominators of the right-hand side indicate that the physical amplitude of reaction (1) is understood as the limiting value of the function A_1 as s_i tends to the real axis from above.

Let us carry out this limiting transition, using the symbolic identity

$$\frac{1}{x' - x - i\varepsilon} = \mathcal{P}\frac{1}{x' - x} + i\pi\delta(x' - x),$$

where $\mathcal{P}$ is the symbol of the principal value. As a result we have:

$$A_1(s_1 s_2 s_3) = A_1(s_1^0 s_2^0 s_3^0) + i\sigma_1(s_1) + i\sigma_2(s_2) + i\sigma_3(s_3) + \\ + \sum_{l=1}^{3} \frac{s_l - s_l^0}{\pi} \int \frac{\sigma_l(s')}{(s' - s_l^0)} \mathcal{P} \frac{ds'}{s' - s_l}. \tag{5}$$

The unitarity condition for the S-matrix leads to the following expressions for the functions σ_1, σ_2, and σ_3:

$$\sigma_1(s_1) = \frac{k_{23}}{\sqrt{k_{23}^2 + \mu^2} + \sqrt{k_{23}^2 + M^2}} A_1 t_S^+(\pi^- n \to \pi^- n); \tag{6a}$$

$$\sigma_2(s_2) = \frac{k_{13}}{\sqrt{k_{13}^2 + \mu^2} + \sqrt{k_{13}^2 + M^2}} \{A_1 t_S^+(\pi^+ n \to \pi^+ n) + A_3 t_S^+(\pi^+ n \to \pi^0 p)\}; \tag{6b}$$

$$\sigma_3(s_3) = \frac{k_{12}}{2\sqrt{k_{12}^2 + \mu^2}} \{A_1 t_S^+(\pi^- \pi^+ \to \pi^- \pi^+) + \frac{1}{2} A_2 t_S^+(\pi^- \pi^+ \to \pi^0 \pi^0)\}. \tag{6c}$$

Here A_i is the amplitude of reaction (i); since we take into account only terms linear in the momenta, while the expressions for σ contain the factor k, in formulas (6) the A_i should be taken as the threshold values of the amplitudes. k_{li} is the modulus of the relative momentum of particles l and i in their center-of-mass system; k_{li} is determined from the relations

$$2k_{li}^2 + m_l^2 + m_i^2 + 2\sqrt{k_{li}^2 + m_l^2}\sqrt{k_{li}^2 + m_i^2} = s_r$$

($r \neq l$, $r \neq i$, m_i is the mass of particle i; the numbering of the particles corresponds to the numbering of the momenta P). $t_S^+(ab \to cd)$ is the S-wave scattering amplitude $a + b \to c + d$.

Attention is drawn to the circumstance that the integral terms in (5) are of second order in the momenta ($s - s^0 \sim k^2$). Neglecting these terms, we obtain:

$$A_1(s_1 s_2 s_3) = A_1(s_1^0 s_2^0 s_3^0) + i[\sigma_1(s_1) + \sigma_2(s_2) + \sigma_3(s_3)], \tag{7}$$

where the σ_i are defined by equalities (6). (Of course, only the terms of the functions σ that are linear in k need be taken into account.)

Analogous expressions can be obtained for reactions (2) and (3). Calculating with the aid of (7) the squares of the moduli of the matrix elements (1), (2), and (3), we arrive at the expressions obtained in work (2) from quantum-mechanical considerations.

Formulas (7) determine not only the modulus of the matrix element, but also the phase of the latter, which is their advantage. (7) is also applicable when the S-wave $\pi\pi$-scattering has a resonance near threshold. In this case it is necessary to retain in (7) the dependence of the amplitudes $t_s^+(\pi\pi \to \pi\pi)$ on the momentum k. Thus, the representation (4) automatically leads to correct expressions for the matrix elements, accurate up to terms of first order of smallness in the relative momenta.

It seems tempting to obtain the terms quadratic in the momenta of the amplitudes of reactions (1), (2), and (3). In doing so, it is already necessary to take into account the dependence of the matrix element on the variables u. But the coefficients of the second powers of the momenta are, roughly speaking, inversely proportional to the distance to the singularities of the amplitude in the corresponding variable. Therefore one may hope that the principal among the quadratic terms are the integrals on the right-hand side of (5). They can be found approximately by substituting into them the first approximation for the functions σ. Apparently, however, such a procedure will make sense only with simultaneous allowance for the P-wave of $\pi\pi$ scattering, not included in the representation (4).

The authors express their gratitude to A. A. Logunov and L. D. Solov' ev for a very useful discussion.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
23 II 1961

References

1. S. Mandelstam, *Phys. Rev.*, **112**, No. 4 (1958).
2. V. N. Gribov, *Nucl. phys.*, **5**, 653 (1958).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.