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Abstract

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ON THE PHENOMENON OF CONVERGENCE IN PERIODIC NONLINEAR SYSTEMS

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Consider the system of differential equations

$$\frac{dx}{dt} = f(x,t), \tag{1}$$

where $x = \{x^{(1)}, \dots, x^{(n)}\}$ and $f(x,t) = \{f^{(1)}(x,t), \dots, f^{(n)}(x,t)\}$ are n-dimensional vectors. It is assumed with respect to $f(x,t)$ that it is continuous, satisfies the condition of uniqueness of solutions of system (1) for all x, t, and has period ω in t for all x: $f(x,t+\omega) = f(x,t)$.

Let $\xi = \{\xi^{(1)}, \dots, \xi^{(n)}\}$ be an n-dimensional vector; put

$$\|\xi\| = \sum_{i=1}^{n} |\xi^{(i)}|.$$

Denote by $x(t, x_0, t_0)$ the solution of system (1) with initial data t_0, x_0.

We shall say that system (1) possesses the property of convergence if:

I. All solutions $x(t, x_0, t_0)$ can be continued for all times $t \geq t_0$.

II. System (1) has a unique ω-periodic solution $x = \varphi(t)$.

III. This solution is stable in the sense of Lyapunov.

IV. For any solution $x(t, x_0, t_0)$ the relation

$$\lim_{t\to+\infty} \|x(t, x_0, t_0) - \varphi(t)\| = 0 \tag{2}$$

holds.

Suppose that all solutions $x(t, x_0, t_0)$ can be continued for $t \geq t_0$. To the point x_0 assign the point $x(\omega, x_0, 0)$. In this way we obtain a homeomorphic transformation T of the space E_n into itself.

We next establish conditions necessary and sufficient for the existence of convergence. These conditions are imposed on the transformation T and do not concern the behavior of the solutions for $0 < t < \omega$. In this they substantially generalize the conditions formulated in the paper (^{1}).

Theorem. *In order that system (1) possess the property of convergence, it is necessary and sufficient that the following conditions be satisfied:*

I. *All solutions $x(t, x_0, t_0)$ can be continued for $t \geq t_0$.*

II. *There exists a closed, bounded set $F \subset E_n$ which is mapped by the transformation T into itself.*

III. *There exists a continuous function $v(x, y)$ of points x and y of the space E_n with the following properties: 1) $v(x, y) \geq 0$ and $v(x, y) = 0$ if and only if $x = y$; 2) for every $x \in F$, $v(x, y) \to \infty$ as $\|y\| \to \infty$; 3) $v(T(x), T(y)) \leq v(x, y)$, and if $v(T(x), T(y)) = v(x, y)$, then $x = y$.*

We outline the proof.

Sufficiency. We shall prove that the transformation T has a fixed point. Consider the set

$$\Omega = \prod_{m=0}^{\infty} T^m(F).$$

It is not difficult to see that Ω is closed and $T(\Omega) = \Omega$. We shall show that Ω is a point of the space E_n. Suppose, on the contrary, that this is not so; then

$$\alpha = \sup_{x, y \in \Omega} v(x, y) > 0. \tag{3}$$

From the closedness of Ω it follows that there exist $\bar{x} \in \Omega$ and $\bar{y} \in \Omega$ such that $v(\bar{x}, \bar{y}) = \alpha$. From the equality $T(\Omega) = \Omega$ it follows that $T^{-1}(\bar{x}) \in \Omega$ and $T^{-1}(\bar{y}) \in \Omega$; by virtue of property (3) of the function $v(x, y)$ we then conclude that

$$v(T^{-1}(\bar{x}), T^{-1}(\bar{y})) > v(\bar{x}, \bar{y}) = \alpha. \tag{4}$$

This contradicts relation (3). Consequently, the set Ω is a point $\xi \in E_n$. Obviously, $T(\xi) = \xi$.

The function $x(t, \xi, 0)$ is an ω-periodic solution of system (1). We shall show that it is unique. Indeed, if there existed a different ω-periodic solution $x(t, \eta, 0)$, $\xi \neq \eta$, then we would have $v(\xi, \eta) > v(T(\xi), T(\eta)) = v(\xi, \eta)$. The contradiction obtained proves the uniqueness of the solution $x(t, \xi, 0)$.

We shall show that the solution $x(t, \xi, 0)$ is Lyapunov stable. Let $\varepsilon > 0$. For this ε, by the theorem on integral continuity, there is a $\delta_1 > 0$ such that if $\|\xi - y\| \leqslant \delta_1$, then

$$\|x(t, y, 0) - x(t, \xi, 0)\| > \varepsilon \quad \text{for } 0 \leqslant t \leqslant \omega. \tag{5}$$

Put

$$\inf_{\delta_1 \leqslant \|\xi - y\| \leqslant \varepsilon} v(\xi, y) = l.$$

By the continuity of v there is a $\delta > 0$ such that, when $\|\xi - y\| \leqslant \delta$, $v(\xi, y) < l$. Now take y such that $\|\xi - y\| \leqslant \delta$. We shall prove that for all $n \geqslant 0$ one has

$$\|\xi - T^n(y)\| < \delta_1. \tag{6}$$

We prove inequality (6) for $n = 1$. If (6) for $n = 1$ were not satisfied, then $\|\xi - T(y)\| \in [\delta_1, \varepsilon]$, which follows from (5). But then $v(\xi, T(y)) \geqslant l$. By the same property 3) of the function v, we have $v(\xi, T(y)) \leqslant v(\xi, y) < l$. This contradiction proves (6) for $n = 1$. For $n > 1$, inequality (6) is established by induction. From relations (5) and (6), and from the periodicity of $f(x, t)$, it follows that the solution $x(t, \xi, 0)$ is Lyapunov stable.

We shall prove that, for arbitrary x_0, t_0,

$$\lim_{t \to +\infty} \|x(t, x_0, t_0) - x(t, \xi, 0)\| = 0. \tag{7}$$

Since all solutions $x(t, x_0, t_0)$ are continuable for $t \geqslant t_0$, to prove (7) it is sufficient to establish the relation

$$\lim_{k \to \infty} T^k(x_0) = \xi. \tag{8}$$

From property 2) of the function v it follows that the set

$$G\{v(\xi, x) \leqslant v(\xi, x_0)\}$$

is bounded. Denote by S the set

$$\prod_{m=0}^{\infty} T^m(G).$$

Just as in the proof of the existence of the fixed point ξ, we establish that S is a fixed point of the transformation T. From the uniqueness of ξ it follows that

$S = \xi$. Clearly, all limit points of the sequence $T^k(x_0)$ are contained in S. Hence (8) follows. The sufficiency of the conditions of the theorem is established.

Necessity. Put

$$\rho(t, x_0, y_0, t_0) = \sum_{k=1}^{n} \left[x^{(k)}(t, x_0, t_0) - x^{(k)}(t, y_0, t_0)\right]^2.$$

Form the function

$$w(x, y, t) = \int_t^\infty G(\rho(\tau, x, y, t))\, d\tau, \tag{9}$$

where $G(z)$ is some function defined for $z \geq 0$, positive for $z > 0$, and vanishing for $z = 0$.

Following the ideas of Massera (^{2}), the function G can be chosen so that the integral on the right-hand side of (9) will converge uniformly for all $\|x\| \leq A,\ \|y\| \leq A,\ 0 \leq t \leq A$, where $A > 0$ is a constant. Thus the function $w(x, y, t)$ is defined and continuous for all x, y and for $t \geq 0$.

Let us show that $w(x, y, t)$ is ω-periodic in the argument t. We have

$$w(x, y, t+\omega) = \int_{t+\omega}^\infty G(\rho(\tau, x, y, t+\omega))\, d\tau = \int_\infty^t G(\rho(\tau+\omega, x, y, t+\omega))\, d\tau. \tag{10}$$

But from the ω-periodicity of $f(x, t)$ it is not difficult to derive the relation

$$\rho = (\tau + \omega, x, y, t + \omega) = \rho(\tau, x, y, t),$$

and therefore from (10) we obtain

$$w(x, y, t + \omega) = \int_t^\infty G(\rho(\tau, x, y, t))\, d\tau = w(x, y, t). \tag{11}$$

We now show that along solutions of system (1) $w(x, y, t)$ decreases. We have

$$\rho(\tau, x(t, x_0, t_0), x(t, y_0, t_0), t) = \rho(\tau, x_0, y_0, t_0). \tag{12}$$

Substituting (12) into (8) and differentiating with respect to t, we obtain

$$\frac{d}{dt} w(x(t, x_0, t_0), x(t, y_0, t_0), t) = -G(\rho(t, x_0, y_0, t_0)).$$

Hence, and from the property of the function $G(z)$, it follows that if $x_0 \neq y_0$, then for all t one has

$$\frac{d}{dt} w(x(t, x_0, t_0), x(t, y_0, t_0), t) < 0. \tag{13}$$

Thus, along solutions of system (1), w decreases.

The function G can be chosen so that for any x satisfying the condition $\|x\| \leq B$, where $B > 0$ is arbitrary but fixed, one has

$$w(x, y, 0) \to \infty \quad \text{as } \|y\| \to \infty. \tag{14}$$

Now put $v(x, y) = w(x, y, 0)$. It is not difficult to see that then all the conditions of the theorem will be fulfilled.

As an example, consider the system

$$\frac{dx_k}{dt} = \sum_{\nu=1}^{n} b_{k\nu} x_\nu + h_k f(\sigma) + p_k(t), \qquad \sigma = \sum_{i=1}^{n} \alpha_i x_i \quad (k = 1, \dots, n). \tag{15}$$

With respect to $f(\sigma)$, it is assumed that it is differentiable for all σ and that $f'(\sigma) > 0$. The functions $p_k(t)$ are assumed to be continuous for all t and $p_k(t + \omega) = p_k(t)$. Suppose that system (15) has the following canonical form (3):

$$\frac{dz_k}{dt} = \lambda_k z_k + f(y) + q_k(t), \qquad y = \sum_{i=1}^{n} \gamma_i z_i \quad (k = 1, \dots, n), \tag{16}$$

where all λ_k have negative real parts and among them there are s real ones and $\frac{1}{2}(n - s)$ pairs of complex conjugates. Suppose further that the system of quadratic equations

$$-2a_k \sum_{i=1}^{n} \frac{a_i}{\lambda_i + \lambda_k} + \gamma_k = 0 \quad (k = 1, \dots, n) \tag{17}$$

has s real roots and $\frac{1}{2}(n-s)$ pairs of complex conjugate roots. Under fulfillment of these conditions, system (15) possesses the property of convergence.

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Note: Figure translations are in progress. See original paper for figures.

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