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Abstract

Full Text

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ON A CLASS OF POLYNOMIALS LEAST DEVIATING FROM ZERO ON TWO INTERVALS

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1. The aim of the present note is to find and investigate polynomials with fixed leading coefficient that deviate least from zero on the two intervals $[-1, -\lambda]$ and $[\lambda, 1]$ with weight $q(x) = \sqrt{1-x^2}$. Polynomials of this kind, but for the constant weight $q(x) = 1$, were constructed and studied by N. I. Akhiezer (^{1,2}) and are a generalization to the case of two intervals of the Chebyshev polynomials of the first kind $T_n(x)$. In contrast to them, the polynomials considered below are the corresponding generalization of the Chebyshev polynomials of the second kind $U_n(x)$. Following the indicated analogy, we shall call them Akhiezer polynomials of the second kind and denote them by $B_n(x, \lambda)$. Here the polynomials $B_n(x, \lambda)$ will be defined as polynomials least deviating from zero on the two intervals $[-1, -\lambda]$ and $[\lambda, 1]$ with weight $\sqrt{1-x^2}$, under the condition that

$$\max_{x \in [\pm\lambda, \pm 1]} \left| \sqrt{1-x^2} B_n(x, \lambda) \right| = 1.$$

Passing to the construction of the polynomials $B_n(x, \lambda)$, we note that the cases of even and odd n differ essentially.

I. The case of even $n = 2m$. In this case the required polynomial is constructed in an elementary way. Indeed, consider the function

$$y_{2m}(u) = \sqrt{1-u^2}\, U_{2m}(u), \quad -1 \le u \le +1, \tag{1}$$

where $U_{2m}(u)$ is the Chebyshev polynomial of the second kind of degree $2m$, normalized in such a way that

$$\max_{u \in [-1,1]} \left| \sqrt{1-u^2}\, U_{2m}(u) \right| = 1.$$

The function $y_{2m}(u)$ is extremal on the segment $[-1, 1]$ in the class of functions of the form $\sqrt{1-u^2}\, P_{2m}(u)$, where $P_{2m}(u)$ is a polynomial of degree $2m$.

Using the transformation

$$u = \sqrt{\frac{x^2 - \lambda^2}{1 - \lambda^2}}, \tag{2}$$

we obtain

$$B_{2m}(x, \lambda) = \frac{1}{\sqrt{1 - \lambda^2}} U_{2m} \left(\sqrt{\frac{x^2 - \lambda^2}{1 - \lambda^2}} \right). \tag{3}$$

For the limiting values $\lambda = 0$ and $\lambda = 1$ we have

$$B_{2m}(x, 0) = U_{2m}(x); \tag{4}$$

$$B_{2m}(x, 1) = (1 - x^2)^m. \tag{5}$$

By means of polynomial (3) the following is solved:

Problem 1. Among all polynomials of degree $2m$ with leading coefficient equal to unity, find the one that deviates least from zero on the two intervals $[-1, -\lambda]$ and $[\lambda, 1]$ with weight $\sqrt{1 - x^2}$.

The required polynomial will be

$$\dot{B}_{2m}(x, \lambda) = \frac{L_{2m}}{\sqrt{1 - \lambda^2}} U_{2m} \left(\sqrt{\frac{x^2 - \lambda^2}{1 - \lambda^2}} \right), \tag{6}$$

where the corresponding minimal deviation L_{2m} is determined by the expression

$$L_{2m} = \frac{(1 - \lambda^2)^{m+1/2}}{2^{2m}}. \tag{7}$$

II. The case of odd $n = 2m - 1$. Let us first set up the differential equation for the function

$$y_{2m-1}(x) = \sqrt{1 - x^2}\, B_{2m-1}(x, \lambda). \tag{8}$$

In accordance with the general theory, the function $y_{2m-1}(x)$ must, on each of the segments $[-1, -\lambda]$ and $[\lambda, 1]$, attain m times its maximum value, equal to unity, successively changing sign. Consequently, the polynomial

$$Q_{4m}(x) = [y_{2m-1}(x)]^2 - \frac{1}{2} \tag{9}$$

must satisfy the equation

$$\frac{1/4-[Q_{4m}(x)]^2}{(1-x^2)(x^2-\gamma^2)(x^2-\lambda^2)}=\frac{[Q'_{4m}(x)]^2}{(4m)^2(x^2-\delta^2)^2},\qquad 0<\gamma\le\delta\le\lambda<1,\tag{10}$$

since the function $1/4-[Q_{4m}(x)]^2$ has simple zeros at the points $x=\pm\gamma$, $x=\pm\lambda$, $x=\pm1$, and also double zeros at $4m-3$ points of the interval $[-1,1]$, where $Q'_{4m}(x)$ also has simple zeros. Moreover, $Q_{4m}(x)$ has two more simple zeros at the points $x=\pm\delta$, where the function $Q_{4m}(x)$ attains an absolute maximum on the segment $[-1,1]$.

As a result we obtain the equation

$$\frac{1-[y_{2m-1}(x)]^2}{(1-x^2)(x^2-\gamma^2)(x^2-\lambda^2)}=\frac{[y'_{2m-1}(x)]^2}{(2m)^2(x^2-\delta^2)^2},\tag{11}$$

whose solution we obtain in the form

$$y_{2m-1}(x)=\sin[2m\,\varphi(x)];\tag{12}$$

$$\varphi(x)=\int_{-1}^{x}\frac{(\delta^2-t^2)\,dt}{\sqrt{(1-t^2)(\gamma^2-t^2)(\lambda^2-t^2)}},\tag{13}$$

where the parameters γ and δ are determined by means of the conditions $y_{2m-1}(0)=0$, $y_{2m-1}(\pm\lambda)=\pm1$, and $y_{2m-1}(\pm\gamma)=\pm1$. In practice it is more convenient to pass to an expression for $y_{2m-1}(x)$ in parametric form through elliptic functions. Following the method of N. I. Akhiezer (^{1}), putting

$$x^2=\frac{\operatorname{sn}^2(K/2m)\operatorname{cn}^2u}{\operatorname{sn}^2(K/2m)-\operatorname{sn}^2u};\qquad \operatorname{sn}\frac{K}{2m}=\lambda;\qquad \lambda\sqrt{\frac{1-k^2}{1-k^2\lambda^2}}=\gamma\tag{14}$$

and taking into account that the function $y_{2m-1}(x)$ has only two poles of multiplicity $2m-1$, corresponding to $x=\pm\infty$, we find

$$\varphi(u)=\frac{i}{2}\ln\frac{H(K/2m+u)}{H(K/2m-u)};\tag{15}$$

$$\delta=\left[\lambda\sqrt{\frac{1-\lambda^2}{1-k^2\lambda^2}}\,\frac{H_1'(K/2m)}{H_1(K/2m)}\right]^{1/2},\tag{16}$$

where $H(u)$ and $H_1(u)$ are theta-functions.

Finally we have

$$y_{2m-1}(x) = \frac{i}{2}\left\{\left[\frac{H(K/2m+u)}{H(K/2m-u)}\right]^m - \left[\frac{H(K/2m-u)}{H(K/2m+u)}\right]^m\right\}, \tag{17}$$

$$B_{2m-1}(x,\lambda) = \frac{i}{2}\sqrt{\frac{\operatorname{sn}^2 u-\lambda^2}{\operatorname{sn}^2 u-\lambda^2\operatorname{sn}^2 u}}\left\{\left[\frac{H(K/2m+u)}{H(K/2m-u)}\right]^m - \left[\frac{H(K/2m-1)}{H(K/2m+u)}\right]^m\right\}; \tag{18}$$

$$\operatorname{sn} u = \lambda\sqrt{\frac{1-x^2}{\lambda^2-x^2}}; \qquad \operatorname{sn}\frac{K}{2m} = \lambda. \tag{19}$$

It should be noted that the function (17) is a solution of equation (11), which is also satisfied by the odd Akhiezer polynomials of the first kind $A_{2m-1}(x,\lambda)$ (^{3}).

For $0 < k < 1$, formulas (17) and (18) determine the extremal functions for $1 > \lambda > \sin(\pi/4m)$. For $0 \le \lambda \le \sin(\pi/4m)$ we obtain

$$y_{2m-1}(x) = \sin(2m\arccos x) = \sqrt{1-x^2}\,U_{2m-1}(x); \qquad B_{2m-1}(x,\lambda) = U_{2m-1}(x), \tag{20}$$

and for $\lambda = 1$

$$B_{2m-1}(x,1) = x(1-x^2)^{m-1}. \tag{21}$$

With the aid of the polynomials $B_{2m-1}(x,\lambda)$ the following is solved:

Problem 2. Among all polynomials of degree $2m-1$ with leading coefficient equal to one, find the one that deviates least from zero on the two intervals $[-1,-\lambda]$ and $[\lambda,1]$ with weight $\sqrt{1-x^2}$.

Taking (18) into account, the required polynomial will be

$$\dot{B}_{2m-1}(x,\lambda) = \frac{iL_{2m-1}}{2}\sqrt{\frac{\operatorname{sn}^2 u-\lambda^2}{\operatorname{sn}^2 u-\lambda^2\operatorname{sn}^2 u}}\left\{\left[\frac{H(K/2m+u)}{H(K/2m-u)}\right]^m - \right.$$
$$\left. - \left[\frac{H(K/2m-u)}{H(K/2m+u)}\right]^m\right\}, \tag{22}$$

where the minimal deviation L_{2m-1} is determined by the relation

$$L_{2m-1} = \frac{1}{2^{2m-1}}\left[\frac{\theta(0)\,\theta_1(0)}{\theta(K/2m)\,\theta_1(K/2m)}\right]^{2m}; \tag{23}$$

here $\theta(u)$ and $\theta_1(u)$ are the corresponding theta-functions.

2. By using the polynomials $B_{2m-1}(x, \lambda)$ constructed above, the following is easily solved:

Problem 3. Among all trigonometric polynomials of the form

$$P_m(\varphi) = a_m \sin m\varphi + a_{m-1} \sin(m-1)\varphi + \ldots + a_1 \sin \varphi \tag{24}$$

with given coefficient a_m, find the one that deviates least from zero on the two intervals $[-\pi, -\varphi_0]$ and $[\varphi_0, \pi]$.

Indeed, putting $x = \sin(\varphi/2)$, we obtain

$$y_{2m-1}(x) = P_m(2 \arcsin x) = \sqrt{1-x^2} \sum_{k=1}^{m} (-1)^{k-1} a_k U_{2k-1}(x); \tag{25}$$

$$-1 \leq x \leq -\lambda; \qquad \lambda \leq x \leq 1; \qquad \lambda = \sin \frac{\varphi_0}{2}.$$

Consequently, the extremal function (25) is determined from the condition

$$\sum_{k=1}^{m} (-1)^{k-1} a_k U_{2k-1}(x) = (-1)^{m-1} l_m \dot{B}_{2m-1}(x, \lambda). \tag{26}$$

Equating the coefficients of x^{2m-1}, we find

$$l_m = 2^{2m-1} a_m L_{2m-1} = a_m \left[\frac{\theta(0)\theta_1(0)}{\theta(K/2m)\theta_1(K/2m)} \right]^{2m}. \tag{27}$$

Thus, the required trigonometric polynomial will be

$$P_m(\varphi, \varphi_0) = (-1)^{m-1} l_m \cos \frac{\varphi}{2} \, B_{2m-1} \left(\sin \frac{\varphi}{2}, \sin \frac{\varphi_0}{2} \right), \tag{28}$$

where l_m is the minimum deviation, determined by formula (27). For the coefficients a_k we have

$$\begin{aligned} a_k &= (-1)^{k-1} \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2} \, B_{2m-1}(x, \lambda) \, U_{2k-1}(x) \, dx \\ &= (-1)^{k-1} k \sum_{p=k}^{m} \frac{b_p C_{2(p-k)}^{k}}{2^{2p-1} p}, \end{aligned} \tag{29}$$

where b_p are the coefficients of the polynomial $B_{2m-1}(x, \lambda)$.

Expression (28) determines the extremal polynomial for $\pi/2m < \varphi_0 < \pi$. For $0 \le \varphi_0 \le \pi/2m$, in accordance with (20), we have

$$P_m(\varphi, \varphi_0) = (-1)^{m-1} l_m \cos\frac{\varphi}{2} U_{2m-1}\left(\sin\frac{\varphi}{2}\right) = a_m \sin m\varphi, \tag{30}$$

and for $\varphi_0 = \pi$

$$P_m(\varphi, \pi) = 2^{2m-1} a_m \sin\varphi (1 - \cos\varphi)^{m-1}. \tag{31}$$

In conclusion we note that the corresponding problem for the polynomial

$$Q_m(\varphi) = a_m \cos m\varphi + a_{m-1} \cos(m-1) + \cdots + a_0$$

is solved in an analogous way, but with the use of the polynomial

$$A_{2m}(x, \lambda) = T_{2m}\left(\sqrt{\frac{x^2 - \lambda^2}{1 - \lambda^2}}\right),$$

found by N. I. Akhiezer in work (1).

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Note: Figure translations are in progress. See original paper for figures.

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