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Abstract

Full Text

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On the Behavior as $t \rightarrow \infty$ of Solutions of a Problem of S. L. Sobolev

(Presented by Academician S. L. Sobolev on 21 III 1961)

Consider the equation

$$\frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial^2 u}{\partial y^2} = 0 \quad (1)$$

and the conditions

$$u|_{t=0} = u_0(x, y); \quad \frac{\partial u}{\partial t} \Big|_{t=0} = u_1(x, y); \quad u(x, y, t)|_{\Gamma} = 0, \quad (2)$$

where Γ is the boundary of a closed bounded domain Ω in the xy -plane. In what follows it is always assumed that the equation of the contour Γ in polar coordinates has the form $x = x(\varphi)$, $y = y(\varphi)$, where $x(\varphi)$, $y(\varphi)$ are three times continuously differentiable, periodic functions with period 2π , satisfying the condition:

$$x'y'' - x''y' \geq q > 0, \quad (3)$$

where q is a certain constant.

Lemma 1. There exists a function $f_1(\varphi, \alpha)$, satisfying the equation $x(f_1) - \text{ctg } \alpha y(f_1) = x(\varphi) - \text{ctg } \alpha y(\varphi)$, defined for $-\infty < \varphi < +\infty$, $-\infty < \alpha < +\infty$, twice continuously differentiable, such that

$$\begin{aligned} \partial f_1(\varphi, \alpha) / \partial \varphi &\leq -q_1 < 0, \\ \partial f_1(\varphi, \alpha) / \partial \alpha &\geq q_1 > 0, \text{ where } q_1 \text{ is a certain constant.} \end{aligned}$$

Set $\xi_n = f_n(\varphi, \alpha) = f_1(\xi_{n-1}, (-1)^{n+1}\alpha)$, $\xi_0 = \varphi$.

Lemma 2. The function $f_n(\varphi, \alpha)$ is a twice continuously differentiable function of φ and α , and moreover

$$\begin{aligned} |\partial f_k(\varphi, \alpha) / \partial \alpha| &\geq q_1, \\ |\partial f_k(\varphi, \alpha) / \partial \varphi| &\geq q_2, \end{aligned}$$

while the function $f_{2N}(\varphi, \alpha) = f_{2N}(\varphi, \alpha) + 4N\alpha - \varphi$ is periodic in α and φ .

Lemma 3. There exist functions $\alpha_{N,k}(\varphi)$ satisfying the conditions:

$$\text{a) } f_{2N}(\varphi, \alpha_{N,k}(\varphi)) = \varphi + 2k\pi;$$

- b) $\alpha_{N,k}(\varphi)$ are periodic functions with period 2π , twice continuously differentiable;
- c) if

$$x(\psi) \pm \operatorname{ctg} \alpha_{N,k}(\psi)y(\psi) = x(\varphi) \pm \operatorname{ctg} \alpha_{N,k}(\varphi)y(\varphi),$$
 then $\alpha_{N,k}(\varphi) = \alpha_{N,k}(\psi)$;
- d) if $k = 2Nl + i$, then $\alpha_{N,k}(\varphi) = \alpha_{N,i}(\varphi) - 2l\pi$.

The functions $f_k(\varphi, \alpha)$, for $k = 1, 2, \dots, 2N$, are the polar angles of the vertices of a certain broken line inscribed in the contour Γ . Let its vertices be $T_k(\varphi, \alpha)$: $T_0(\varphi)$ coincides with the point corresponding to the polar angle φ . Denote by $L_{k,k+1}(\varphi, \alpha)$ the straight line joining the points $T_k(\varphi, \alpha)$, $T_{k+1}(\varphi, \alpha)$. In ⁽²⁾ the existence is shown, for each N , of an angle $\alpha_N(\varphi)$ such that $T_{2N}(\varphi, \alpha_N(\varphi)) = T_0(\varphi)$, i.e. the broken line arrives at the point from which it started. This result is used in ⁽²⁾ for constructing, in certain domains, differential solutions of the operator $A = \Delta^{-1}\partial^2/\partial y^2$, Δ^{-1} restoring

function $v, v|_{\Gamma} = 0$, from the given $\Delta v = \partial^2 v/\partial x^2 + \partial^2 v/\partial y^2$, whence follows the existence of non-almost-periodic solutions of problem (1)–(2). We must study the differential properties of the functions $\alpha_{N,k}(\varphi)$, and therefore Lemmas 1–3 differ somewhat from those given in ⁽²⁾.

The proof of the following lemma, due to P. A. Aleksandryan, can be found in ⁽²⁾.

Lemma 4. Let a_1, a_2 be such that $T_{2N}(\varphi_1, a_1) = T_0(\varphi_1)$, $T_{2N}(\varphi_2, a_2) = T_0(\varphi_2)$, $|a_1 - a_2| < \varepsilon$, $|\varphi_1 - \varphi_2| < \varepsilon$, $\varepsilon > 0$ and sufficiently small; then $L_{k,k+1}(\varphi_1, a_1)$ and $L_{k,k+1}(\varphi_2, a_2)$ do not intersect inside the domain Ω .

From Lemma 4 it follows that $L_{k,k+1}(\varphi, \alpha_{N,i}(\varphi))$ and $L_{k,k+1}(\varphi', \alpha_{N,i}(\varphi'))$ cannot intersect inside Ω for any φ' , $T_0(\varphi) \neq T_0(\varphi')$. Using Lemmas 1–4, one can prove the following lemma:

Lemma 5. There exist functions $\mu_{N,k,i}(x, y)$, $N = 2, 3, \dots, \infty$, $i = 1, 2$, $k = 1, 2, \dots, 2N$, such that:

- a) $\mu_{N,k,1}(x, y)|_{\Gamma} = \mu_{N,k,2}(x, y)|_{\Gamma} = \lambda_{N,k}(\varphi) = \operatorname{ctg} \alpha_{N,k}(\varphi)$;
- b) $\mu_{N,k,i} \partial \mu_{N,k,i} / \partial x = (-1)^i \partial \mu_{N,k,i} / \partial y$;
- c) $\mu_{N,k,i}(x, y)$ are uniquely defined at every point $x, y \in \Omega$, and twice continuously differentiable at all points of $\bar{\Omega}$, except for the points lying on the straight line $y = y(\varphi')$, where $\bar{\alpha}_{N,k}(\varphi') = 0$, if such φ' exist.

Lemma 6. Any solution of equation (1) of the form $u = v(x, y)e^{i\tau(x,y)t}$ with $\partial\tau/\partial x \neq 0$ can be obtained for $v(x, y) = \frac{\partial\mu}{\partial y}f(\mu)$, $\tau = \pm\sqrt{\mu^2/(1+\mu^2)}$, where f is some twice differentiable function, $\mu^2(\partial\mu/\partial x)^2 = (\partial\mu/\partial y)^2$.

It is easy to see that the function

$$\begin{aligned}
 u(x, y) &= \frac{\partial}{\partial y} \int_{\mu_0}^{\mu(x, y)} f(\alpha) \exp \left[i \sqrt{\frac{\alpha^2}{1 + \alpha^2}} t \right] d\alpha = \\
 &= \pm \frac{\partial}{\partial x} \int_{\mu_0}^{\mu(x, y)} \alpha f(\alpha) \exp \left[i \sqrt{\frac{\alpha^2}{1 + \alpha^2}} t \right] d\alpha.
 \end{aligned}$$

Theorem 1. The function

$$u_{N, k}(x, y, t) = \int_{\mu_{N, k, 1}(x, y)}^{\mu_{N, k, 2}(x, y)} f(\alpha) \exp \left[i \sqrt{\frac{\alpha^2}{1 + \alpha^2}} t \right] d\alpha, \quad (4)$$

where $f(\alpha)$ is a differentiable finite function, is a solution of problem (1)–(2). If the domain Ω is such that not all $\lambda_{N, k}(\varphi)$ are constant, then $u_{N, k}$ is not identically zero for at least one k when $N = N^*$, $\partial \lambda_{N^*, k} / \partial \varphi \neq 0$, and for $f(\alpha) \neq 0$ in the interval of integration.

It is easy to verify that $u_{N, k}(x, y, t) = f_1(x + \mu_{N, k, 1}y) + f_2(x - \mu_{N, k, 2}y)$ for $t = 0$, where f_1 and f_2 are certain functions.

The same initial functions, for constant $\lambda_{N, k}(\varphi)$ and naturally chosen f_1 and f_2 , generate a periodic solution of the problem. If $\lambda_{N, k}(\varphi)$ is not constant, solutions of the form (4) arise.

Theorem 2. Let $\lambda_{N, k}(\varphi)$ be constant for all N and k . Then the point part of the spectrum of the operator A is everywhere dense in the set formed by the whole spectrum of this operator, considered as an operator in the space of S. L. Sobolev $W_2^1(\Omega)$.

Lemma 7. The function $\alpha_{N, k}(\varphi)$ for each N and k is a continuous function of the domain, i.e., if $x = x_\varepsilon(\varphi)$, $y = y_\varepsilon(\varphi)$ and $\rho_\varepsilon(\varphi) = \sqrt{x_\varepsilon^2 + y_\varepsilon^2} \rightarrow \rho = \sqrt{x_0^2 + y_0^2}$ as $\varepsilon \rightarrow 0$, then $\alpha_{N, k}(\varphi, \varepsilon) \rightarrow \alpha_{N, k}(\varphi, 0)$.

Consider the equation

$$\sum_{i=0}^M \frac{\partial^i}{\partial t^i} \left(a_i \frac{\partial^2 u}{\partial x^2} + b_i \frac{\partial^2 u}{\partial y^2} \right) = 0, \quad (5)$$

where a_i, b_i are constant real numbers. Let $P(\lambda) = \sum a_i \lambda^i$, $Q(\lambda) = \sum b_i \lambda^i$, $\rho_i(\alpha)$ be the solutions of the equation

$$P(\lambda) + \alpha^2 Q(\lambda) = 0. \quad (6)$$

Theorem 3. The function

$$u(x, y, t) = \int_{\mu_{N,k,1}(x,y)}^{\mu_{N,k,2}(x,y)} f(\alpha) \exp[\rho_i(\alpha)t] d\alpha, \quad (7)$$

where $f(\alpha)$ is a differentiable finite function, is a twice continuously differentiable solution of equation (5), and moreover $u|_{\Gamma} = 0$.

The theorem is obvious in the case when all $\rho_k(\alpha)$ are constant, i.e., when for some constant K , $P(\lambda) = KQ(\lambda)$.

In the case when P and Q have no common roots and the functions $\rho_1(\alpha), \dots, \rho_M(\alpha)$ are distinct, the solution of equation (5) with initial data

$$\left. \frac{\partial^i u}{\partial t^i} \right|_{t=0} = 0 \quad \text{for } i \neq j; \quad i = 0, 1, \dots, j-1, j, \dots, M-1;$$

$$\left. \frac{\partial^j u}{\partial t^j} \right|_{t=0} = \int_{\mu_{N,k,1}(x,y)}^{\mu_{N,k,2}(x,y)} f(\alpha) d\alpha \quad (8)$$

can be represented in the form

$$u(x, y, t) = \int_{\mu_{N,k,1}(x,y)}^{\mu_{N,k,2}(x,y)} \sum \varphi_i(\alpha) \exp[\rho_i(\alpha)t] d\alpha, \quad (9)$$

where $\sum \varphi_i(\alpha) \exp[\rho_i(\alpha)t]$ is a differentiable function of its argument α .

As $t \rightarrow \infty$, the functions $u(x, y, t)$ defined by relation (4) decrease at infinity; $\partial u / \partial x$, $\partial u / \partial y$, for fixed x, y , are almost periodic functions of t ; $\partial^2 u / \partial x^2$, $\partial^2 u / \partial y^2$, $\partial^2 u / \partial x \partial y$ grow no faster than t .

The asymptotics of the functions defined by equation (7) depend on the properties of the functions $\rho_i(\alpha)$.

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CITED LITERATURE

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2. R. A. Aleksandryan, Dissertation, Moscow State University, 1949.

Note: Figure translations are in progress. See original paper for figures.

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