Mathematics

G. B. Gurevich

1961

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Abstract

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G. B. Gurevich

On Certain Properties of Metabelian Lie Algebras

(Presented by Academician P. S. Novikov, February 7, 1961)

1. Let $\mathfrak{B}$ be an abstractly given **metabelian** (nilpotent of nilpotency class 2) **Lie algebra** over an algebraically closed field K; let $\mathfrak{A}$ be its **algebra of derivations**. Choosing a Cartan subalgebra Γ of the algebra $\mathfrak{A}$, we define in $\mathfrak{B}$ **weight vectors**

$$a_1, a_2, \dots, a_N, \tag{1}$$

where N is the dimension of the algebra $\mathfrak{B}$; to each vector a_x there corresponds a weight Λ_x, which is a linear form in the parameters of the algebra Γ. The forms $\Lambda_1, \Lambda_2, \dots, \Lambda_N$ are the weights of the algebra $\mathfrak{A}$ on the space of the algebra $\mathfrak{B}$; for brevity we shall call them the **weights of the algebra** $\mathfrak{B}$.

Among the vectors (1) we distinguish weight vectors of the **second** and **first order**, according as they do or do not belong to the commutant $[\mathfrak{B}^2]$. We shall take the indices of weights and weight vectors from the first half of the alphabet $(i, j, \dots)$ if the vectors are of the first order, and from the second half $(p, q, r, \dots)$ if they are of the second order; the indices x, y, z will be reserved for arbitrary weights and weight vectors.

With this notation,

$$[a_i a_p] = [a_p a_q] = 0;$$

$$[a_i a_j] = \theta a_p. \tag{2}$$

To every commutator of the form (2), in which the scalar $\theta \neq 0$, there corresponds a **basic relation**

$$\Lambda_i + \Lambda_j - \Lambda_p = 0 \tag{3}$$

between the weights of the algebra $\mathfrak{B}$.

It can be shown that:

1°. *Every linear dependence between the weights of a Lie algebra* $\mathfrak{B}$ *is a consequence of all the basic relations* (3).

Proposition 1° is valid for any Lie algebra. In the case of metabelian Lie algebras, 1° implies:

2°. *Every weight of a metabelian Lie algebra is different from zero.*

3°. *The weight of a vector of the first order cannot be equal to the weight of a vector of the second order.*

For nonmetabelian Lie algebras, proposition 2° is no longer valid: in (1) an example is given of a nilpotent Lie algebra of nilpotency class 3 all of whose weights are equal to zero.

2. In what follows we consider only those metabelian Lie algebras for which all weights are simple. Metabelian Lie algebras having the same basic relations

$$S_x = 0, \qquad x = 1, 2, \dots, F, \tag{4}$$

between the weights will be grouped into a **family** $\mathfrak{B}$; in (4)

$$S_x = \Lambda_{i_x} + \Lambda_{j_x} - \Lambda_{p_x}. \tag{5}$$

To the fundamental relation $S_x = 0$ there corresponds the commutator

$$[a_{i_x} a_{j_x}] = \theta_x a_{p_x}, \qquad \theta_x \neq 0. \tag{6}$$

As is not difficult to show:

4°. *To each linear dependence*

$$\sum_{x=1}^{F} \alpha_x S_x = 0$$

there corresponds an invariant

$$J = \prod_{x=1}^{F} \theta_x^{\alpha_x}$$

with respect to any renumbering

$$\bar{a}_z = \rho_z a_z, \qquad z = 1, 2, \dots, N, \tag{7}$$

of the weight vectors (1).

3. On the basis of known propositions of the theory of Lie algebras, we establish the validity of the following theorem:

5°. *Let* $\overline{\mathfrak{B}}$ *be a class of metabelian Lie algebras of dimension* N *over an algebraically closed field* K, *defined by* F *fundamental relations* (4) *among the weights* Λ_x $(x = 1, 2, \dots, N)$ *of the algebras of the class; suppose, moreover, that, by virtue of* (4), *all weights* Λ_x *are simple.*

If ρ *is the number of linearly independent relations among* (4) *and* $\rho = F$, *then all algebras of the class* $\overline{\mathfrak{B}}$ *are isomorphic to one another.*

If, however, $\rho < F$ *and the first* ρ *relations* (4) *are linearly independent, then there may be constructed a system of* $F - \rho$ *invariants*

$$J_y = \prod_{x=1}^{y+\rho} \theta_x^{\rho_{yx}}, \qquad y = 1, 2, \dots, F - \rho \tag{8}$$

(ρ_{yx} *are integers), possessing the following property: two Lie algebras of the class* $\overline{\mathfrak{B}}$ *are isomorphic if and only if for them the values of all invariants of the indicated system are respectively either equal, or can be made equal by one of those substitutions on* $a_1, a_2, \dots, a_N$ *which leave unchanged the system* (4) *of fundamental relations among the weights* $\Lambda_1, \Lambda_2, \dots, \Lambda_N$ *of the algebras of the class* $\overline{\mathfrak{B}}$.

Since isomorphic metabelian Lie algebras obviously belong to one and the same class, theorem 5° gives conditions for the isomorphism of two metabelian Lie algebras over an arbitrary algebraically closed field in the case when all weights of both algebras are simple.

The specific structure of the invariants (8) makes it possible to prove also the following proposition:

6°. *Under the conditions of theorem* 5°, *any Lie algebra* $\mathfrak{B}$ *of the class* $\overline{\mathfrak{B}}$ *can, by a renumbering* (7) *of the weight vectors, be brought to a canonical form in which certain fixed* ρ *of the scalars* θ_x $(x = 1, 2, \dots, F)$ *are equal to unity.*

The indicated canonical form thus contains $F - \rho$ arbitrary parameters.

The proofs of propositions 5° and 6° remain valid for arbitrary Lie algebras; however, for non-metabelian Lie algebras the requirement that all weights be simple is excessively burdensome.

I note that the problems of classification of metabelian Lie algebras and of linear systems of bivectors are equivalent to one another; in turn, any ⋯

to the Lie algebra with structure tensor c_{ij}^{k} one may associate a linear system of bivectors $c_{ij}^{k} p_k$, where p_k is an arbitrary covariant vector.

4. A family $\mathfrak{B}$ of metabelian Lie algebras is called **nondegenerate** if every equality of the form $\Lambda_i + \Lambda_j - \Lambda_p = 0$ that is a consequence of the basic relations (4) is contained among them. In what follows only nondegenerate families are considered; moreover, as in §§ 2, 3, we assume that the weights Λ_x are simple. For brevity we shall write a_i, a_{jx}, a_{pz}, etc., mostly as $i, j_x, p_z, \ldots$.

If for a root affinor (linear operator) U of the algebra of differentiations $\mathfrak{A}$ of a Lie algebra $\mathfrak{B} \in \overline{\mathfrak{B}}$ the equality

$$Ua_y = \varphi a_x, \qquad \varphi \neq 0, \tag{9}$$

holds, then, as is known, the root corresponding to the affinor U is $\alpha = \Lambda_x - \Lambda_y$. We shall call the difference $\Lambda_x - \Lambda_y$ **admissible** for the family $\mathfrak{B}$ if it contains at least one algebra $\mathfrak{B}$ for which the relations (9) hold.

We shall mark the weight of a vector a_i of the first queue by an asterisk if a_i belongs to the center of an algebra $\mathfrak{B} \in \overline{\mathfrak{B}}$. In this notation the admissible differences will be all differences of the form

$$\Lambda_p - \Lambda_i, \qquad \Lambda_p - \Lambda_i^*, \qquad \Lambda_i^* - \Lambda_j, \qquad \Lambda_i^* - \Lambda_j^*, \tag{10}$$

as well as differences equal to zero. The first of the differences (10) is associated in the algebra of differentiations of any of the algebras $\mathfrak{B} \in \overline{\mathfrak{B}}$ with an affinor C for which

$$Ci = p, \qquad Ca_x = 0, \qquad x \neq i;$$

the same holds analogously for the remaining differences (10); differences equal to zero correspond to affinors of the Cartan subalgebra Γ of the algebra $\mathfrak{A}$. The differences $\Lambda_i - \Lambda_p$, $\Lambda_i^* - \Lambda_p$, $\Lambda_i - \Lambda_j^*$ are always inadmissible. For a difference of the first queue $\Lambda_i - \Lambda_j$ and of the second queue $\Lambda_p - \Lambda_q$, the question of their admissibility is decided considerably more difficultly (see §§ 5, 6).

5. Two equal differences of the first queue $\Lambda_{i_1} - \Lambda_{j_1}$ and $\Lambda_{i_2} - \Lambda_{j_2}$ will be called **directly connected** if each of the equal sums $\Lambda_{i_1} + \Lambda_{j_2}$ and $\Lambda_{j_1} + \Lambda_{i_2}$ occurs in the left-hand side of one of the basic relations (4) (by virtue of the nondegeneracy of the family this is true either for both sums or for neither of them). Directly connected differences $\Lambda_i - \Lambda_j$ and $\Lambda_p - \Lambda_q$ are defined in a similar way. If one difference is directly connected with a second, the second with a third, and so on, then all these differences are **connected** with one another.

To each of the root affinors $U \in \mathfrak{A}$ there corresponds a separate system of differences of the first and second queues, equal to one another and connected with one another:

$$\begin{gathered}\Lambda_{i_1} - \Lambda_{j_1} = \Lambda_{i_2} - \Lambda_{j_2} = \cdots = \Lambda_{i_s} - \Lambda_{j_s} = \\ = \Lambda_{p_1} - \Lambda_{q_1} = \Lambda_{p_2} - \Lambda_{q_2} = \cdots = \Lambda_{p_t} - \Lambda_{q_t}.\end{gathered} \tag{11}$$

Let I denote the set of indices i_x occurring in (11), and of the corresponding weights and weight vectors; analogously we define the sets J, P, Q. To the system (11) there corresponds an affinor $U \in \mathfrak{A}$ for which

$$Uj_x = \varphi_x i_x, \qquad x = 1, 2, \dots, s;$$

$$Uq_z = \tau_z p_z, \qquad z = 1, 2, \dots, t;$$

$$Ua_y = 0, \qquad y \notin J + Q.$$

The conditions that U belong to $\mathfrak{A}$ lead to a **system (A) of equations** for the scalars φ_x, τ_z. Each of these equations is generated

by two or three directly connected differences; in the first case we shall say that these two differences are joined by an **ordinary connection**, in the second case we have a **triangular connection**. The following assertion is obvious:

7°. *If one of two differences joined by an ordinary connection is inadmissible, then the same is true of the other; if, in a triangular connection, two differences are inadmissible, then the third will also be inadmissible.*

The next two propositions give conditions necessary for admissibility.

8°. *If in (11) an x-difference of the first order is admissible and the weight Λ_{ix} enters into some one of the basic relations (4) in which the weight of the second order is not in P, then in that same relation the second weight of the first order must belong to J.*

9°. *If in (11) a z-difference of the second order is admissible, then in those of the basic relations (4) in which Λ_{qz} participates, at least one of the weights of the first order must belong to J.*

6. The system (A) (Sec. 5) splits into several systems (B); in two systems (B) the coefficients of the unknowns are different. Each of the systems (B) arises when considering **pairs of sequences** of the form:

$$\begin{gathered} j_1 \notin J; \quad i_1 = j_3; \quad i_3 = j_5; \ \dots; \ i_{2g-3} = i_{2g-1}; \quad i_{2g-1} \notin J; \\ j_2 \notin J; \quad i_2 = j_4; \quad i_4 = j_6; \ \dots; \ i_{2h-2} = j_{2h}; \quad i_{2h} \notin J \end{gathered} \tag{12}$$

(the numbering of the differences in (11) has been changed). In view of the nondegeneracy of the rank,

$$[i_{2x-1} i_{2y}] = [i_{2x-1} i_{2y+2}] = [i_{2x+1} i_{2y}] = [i_{2x+1} i_{2y+2}] = \omega_{xy} r_{x+y};$$

$$x = 0, 1, \dots, g; \qquad y = 0, 1, \dots, h.$$

The vectors r_{x+y} determine differences of the second order, which together with the sequences (12) form one or several **chains**. The same vectors determine the numbers d and w on the basis of the relations

$$r_{g+h},\ r_{g+h-1},\ \dots,\ r_{v+1} \notin P; \qquad r_v \in P; \qquad d = g + h - v;$$

$$r_v,\ r_{v-1},\ \dots,\ r_{w+1} \in P; \qquad r_w \notin P.$$

10°. *If $d < w$, then all differences participating in the chains determined by a pair of sequences (12) are inadmissible for $\mathfrak{B}$. The same is true also for $d = w$ (excluding the special case indicated below), if the polynomial*

$$\Phi(x, y) = \sum_{z=0}^{w} (-1)^z C_w^z \frac{(g - w + z)!(h - z)!}{(g - w)!(h - w)!} \frac{x!}{(x - w + z)!} \frac{y!}{(y - z)!}$$

vanishes at least at one of the integral points of the domain

$$0 \leqslant x \leqslant g, \qquad 0 \leqslant y \leqslant h, \qquad w \leqslant x + y \leqslant g + h - d.$$

In the **special case** where $d = w = 1,\ g = h = 2$, three of the six differences are inadmissible.

The theorem below resolves the question of admissibility of differences of the first and second order between the weights of metabelian Lie algebras of nondegenerate rank $\mathfrak{B}$ (under the assumption that all weights Λ_x are simple).

11°. *If we exclude from a separate system (11) those differences which are inadmissible for $\mathfrak{B}$ in view of the failure of the necessary conditions 8°, 9°, or else by virtue of Proposition 10°, and also those which then prove inadmissible by the rule 7°, then all remaining differences (11) will be admissible for the rank $\mathfrak{B}$.*

Moscow State Pedagogical Institute
named after V. I. Lenin

Received
6 II 1961

REFERENCES

1. J. Dixmier, W. G. Lister, *Proc. Am. Math. Soc.*, **8**, 155 (1957).

Note: Figure translations are in progress. See original paper for figures.