V. A. SHCHERBINA

We shall base our considerations on a nonlocal interaction Lagrangian of the form

1961

SovietRxiv

Abstract

Full Text

V. A. SHCHERBINA

THE CAUSALITY CONDITION AND NONLOCALITY OF THE LAGRANGIAN IN QUANTUM ELECTRODYNAMICS

(Presented by Academician N. N. Bogolyubov, 3 I 1961)

We shall base our considerations on a nonlocal interaction Lagrangian of the form

$$\mathcal{L}(x) = e \int \bar{\psi}(x+\xi_1)\hat{A}(x+\eta)\psi(x+\xi_2)F(\xi_1,\xi_2,\eta)d\xi_1 d\xi_2 d\eta. \tag{1}$$

Here $F(\xi_1,\xi_2,\eta)$ is a generalized function on the Schwartz space S. In addition, we require that its Fourier transform $\widetilde{F}(p_1,p_2,k)$ be continuous in a neighborhood of the surface Γ $(p_1^2 = p_2^2 = m^2,\ k^2 = 0)$ and bounded on it. As will be seen below, this requirement can be considerably weakened.

Finally, the last condition on F follows from the self-adjointness of the Lagrangian $\mathcal{L}(x) = \mathcal{L}^*(x)$:

$$F(\xi_1,\xi_2,\eta) = F^*(\xi_2,\xi_1,\eta). \tag{2}$$

If one starts from the known expansion of the scattering matrix S in powers of e, then the Lagrangian $\mathcal{L}$ must also be subjected to the causality condition

$$[\mathcal{L}(x),\mathcal{L}(x')] = 0 \quad \text{for } (x-x')^2 < 0. \tag{3}$$

This is necessary in order to give an invariant meaning to the T-products occurring in the indicated expansion of S.

A consequence of condition (3) and of the conditions on $F(\xi_1,\xi_2,\eta)$ is the following

Theorem. *In order that the Lagrangian $\mathcal{L}(x)$ satisfy the causality condition (3), it is necessary and sufficient that on the surface Γ the equality*

$$\widetilde{F}(p_1,p_2,k) = ce^{i\{\varphi(p_1)+\varphi(p_2)+\tau(k)\}}, \tag{4}$$

hold, where φ,τ are arbitrary odd, continuous functions.

As is easy to see, equality (4) means the unitary equivalence (up to a constant factor) of the Lagrangian (1) to the usual local interaction Lagrangian.

For the proof of the theorem we shall need the following two lemmas.

Lemma 1. *A function $D(x)$ satisfies the Klein-Gordon equation and vanishes outside the light cone if and only if its Fourier transform has the form $\widetilde{D}(k) = P(k)\varepsilon(k^0)\delta(k^2-m^2)$, where $P(k)$ is some polynomial.*

The formulated lemma follows in an obvious way from the expression for the solution of the Cauchy problem for the Klein-Gordon equation. From it there immediately follows the impossibility of smoothing the permutation functions for the field operators $\psi(x)$ and $A(x)$ by subjecting them to a transformation of convolution type: $\bar{u}(x) = \int u(x+\xi)f(\xi)d\xi$.

Lemma 2. If the function $D(x) = \sum_{r=1}^{n} e^{ik_r x} D_r(x)$, where all k_r are distinct and

$$\widetilde{D}_r(k) = \varphi_r(k)\delta(k^2-m^2)$$

(m may also be equal to zero, $\varphi_r(k)$ are bounded on the surface $k^2 = m^2$), vanishes outside the light cone, then also all $D_r(x) = 0$ for $x^2 < 0$.

We shall omit the proof of this lemma because of its unwieldiness.

To prove the theorem, consider the equality

$$\begin{aligned}
&[\bar{\psi}(x+\xi_1)\hat{A}(x+\eta)\psi(x+\xi_2), \bar{\psi}(x'+\xi_1')\hat{A}(x'+\eta')\psi(x'+\xi_2')] \\
&= ig^{mn}\bar{\psi}(x+\xi_1)\gamma^m\psi(x+\xi_2)\bar{\psi}(x'+\xi_1')\gamma^n\psi(x'+\xi_2')D_0(x-x'+\eta-\eta') \\
&\quad - i\hat{A}(x'+\eta')\hat{A}(x+\eta)\bar{\psi}(x+\xi_1)\psi(x'+\xi_2')S(x-x'+\xi_2-\xi_1') \\
&\quad - i\hat{A}(x'+\eta')\hat{A}(x+\eta)\bar{\psi}(x'+\xi_1')\psi(x+\xi_2)S(x'-x+\xi_2'-\xi_1).
\end{aligned} \tag{5}$$

In the right-hand side of equality (5) there are two types of operator polynomials. It is not difficult to see that, putting

$$\Phi_1 = |0\rangle, \qquad \Phi_2 = a^+_{\alpha}(p_1)a^+_{\alpha'}(p_1')a^+_{\beta}(p_2)a^+_{\beta'}(p_2')|0\rangle$$

(all the notation we use is taken from (1)), we obtain

$$\begin{aligned}
\Phi_1[\mathcal{L}(x),\mathcal{L}(x')]\Phi_2 = e^{-i(p_1+p_2+p_1'+p_2')x'}\Big\{&c_1 e^{-i(p_2'+p_1)(x-x')}D(p_2',p_1,p_2,p_1';x-x') \\
&+ c_1 e^{-i(p_2+p_1')(x-x')}D(p_2,p_1',p_2',p_1;x-x') \\
&+ c_2 e^{-i(p_2+p_1)(x-x')}D(p_2,p_1,p_2',p_1';x-x') \\
&+ c_2 e^{-i(p_2'+p_1')(x-x')}D(p_2',p_1',p_2,p_1;x-x')\Big\} = 0
\end{aligned} \tag{6}$$

for $(x-x')^2 < 0$.

Here

$$D(p_1,p_2,p_3,p_4;x-x') = \int D_0(x-x'+\eta)\,F_1(p_1,p_2,p_3,p_4;\eta)\,d\eta,$$

where F_1 is the Fourier transform, with respect to all variables except η, of the function

$$\int F(\xi_1, \xi_2, \eta + \xi) F(\xi_3, \xi_4, \xi)\, d\xi.$$

From (6), on the basis of Lemma 2, we conclude that the function

$$D(p_1, p_2, p_3, p_4; x - x') = 0 \quad \text{for } (x - x')^2 < 0 \tag{7}$$

for arbitrary p_i belonging to the upper sheet of the hyperboloid $p^2 = m^2$. Considering the other mean values of type (6), it is not difficult to show that (7) holds for all p_i belonging to the hyperboloid $p^2 = m^2$.

We now form a mean value of type (6) with the help of the vectors

$$\Phi_1 = |0\rangle, \qquad \Phi_2 = a_m^{(+)}(k) a_n^{(+)}(k') a_\mu^+(p) a_\mu^+(p')|0\rangle.$$

Carrying out calculations entirely analogous to the preceding ones, we obtain that the function

$$\left(i \sum_n c_n \frac{\partial}{\partial x_n} + am \right) D_1(p_1, p_2, k_1, k_2; x - x') = 0 \quad \text{for } (x - x')^2 < 0. \tag{8}$$

Here

$$c_n = \bar{v}^{\nu,-}(p_2) \hat{e}^2 \gamma^n \hat{e}^1 v^{\mu,-}(p_1), \qquad a = \bar{v}^{\nu,-}(p_2) \hat{e}^2 \hat{e}^1 v^{\mu,-}(p_1),$$

and

$$D_1(p_1, p_2, k_1, k_2; x - x') = \int D(x - x' + \xi) F_2(p_1, p_2, k_1, k_2; \xi)\, d\xi.$$

F_2 is the Fourier transform, with respect to all variables except ξ, of the function

$$\int F(\xi_1, \xi + \zeta, \eta_1) F(\zeta, \xi_2, \eta_2)\, d\zeta.$$

Let us prove that

$$D_1(p_1, p_2, k_1, k_2; x) = 0 \quad \text{for } x^2 < 0 \tag{9}$$

for any p_i lying on the hyperboloid $p^2 = m^2$, and k_i on the cone $k^2 = 0$.

Using the relation $\hat{a}\hat{b} + \hat{b}\hat{a} = 2(ab)$, we obtain

$$\hat{e}^2 \gamma^n \hat{e}^1 = 2[e_n^2 e^1 - (e^1 e^2)\gamma^n + e_n^1 e^2] - \hat{e}^1 \gamma^n \hat{e}^2.$$

For $n = 0$ and $e^1 \perp e^2$, $\hat{e}^2\gamma^0\hat{e}^1 = -\hat{e}^1\gamma^0\hat{e}^2$. From this relation it is clear that, by changing the polarization of the photons in the vector Φ_2, we can change the sign of γc_0, while the remaining c_n are transformed differently. Therefore, from (8), on the basis of the first lemma, we conclude that (9) is valid.

From consideration of equality (7) and using (9), with the help of the first lemma we immediately arrive at the conclusion that on the surface Γ the equalities

$$\begin{aligned} \widetilde{F}(p_1, p_2, k)\,\widetilde{F}(p_3, p_4, -k) &= \Phi_1(p_1, p_2)\Phi_1(p_3, p_4), \\ \widetilde{F}(p_1, p_2, k)\,\widetilde{F}^*(p_3, p_4, k) &= \Phi_1(p_1, p_2)\Phi_1^*(p_3, p_4) \end{aligned} \tag{A}$$

hold.

The second of the equalities (A) follows from the first if one takes into account the self-conjugacy condition (2), which in terms of Fourier transforms has the form

$$\widetilde{F}(p_1, p_2, k) = \widetilde{F}^*(-p_2, -p_1, -k).$$

From the written relations it already follows without difficulty that

$$\widetilde{F}(p_1, p_2, k) = e^{i\tau(k)}\Phi_1(p_1, p_2),$$

where $\tau(k)$ is an odd function of k.

In exactly the same way, from equality (9) we obtain for $\Phi_1(p_1, p_2)$ two relations

$$\begin{aligned} \Phi_1(p_1, p)\Phi_1(-p, p_2) &= \Phi_2(p_1)\Phi_3(p_2), \\ \Phi_1(p_1, p)\Phi_1^*(p_2, p) &= \Phi_2(p_1)\Phi_2^*(p_2). \end{aligned} \tag{B}$$

On the basis of relations (B) we conclude that on the surface Γ

$$F(p_1, p_2, k) = ce^{i[\varphi(p_1)+\varphi(p_2)+\tau(k)]},$$

where ψ, φ are odd functions.

Kharkov State University
named after A. M. Gorky

Received
30 XII 1960

CITED LITERATURE

1. N. N. Bogolyubov, D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Moscow, 1957.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.