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Abstract

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ON SOME INTEGRAL RELATIONS IN THE ACOUSTICS OF A MOVING MEDIUM

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In paper (1) it was noted that the reciprocity principle in acoustics can be formulated mathematically in the form of a number of integral relations connecting the solutions of two self-adjoint boundary-value problems. In the acoustics of a moving medium, relations may be obtained that are in a certain sense analogous to reciprocity relations, although, as is known, the reciprocity principle is not satisfied in a moving medium (2). These relations connect the solutions of two adjoint boundary-value problems and establish a connection between volume sources, certain surface forces acting on elastic thin bodies in a moving medium, and the sound fields caused by sources and by vibrations of elastic bodies.

Consider an arbitrary volume Ω of space occupied by a homogeneous gas flow moving with velocity V relative to the chosen coordinate system. Suppose that in Ω there are stationary thin elastic bodies—plates and cylindrical shells, free or fixed in stationary, absolutely rigid screens and arranged in Ω in such a way that the directions of the generators of the shells and plates coincide with the direction of the flow. Let the surface of a shell be denoted by S_i, and the contour of attachment by Γ_i.

Let $p^{(1)}(\mathbf{r})$ be the field produced by some system of harmonic sources $Q^{(1)}(\mathbf{r})$ in Ω. Then $p^{(1)}(\mathbf{r})$ is a solution of the equation

$$\Delta p^{(1)}(\mathbf{r}) - \frac{1}{c^2}\left(-i\omega + V\frac{\partial}{\partial x}\right)^2 p^{(1)}(\mathbf{r}) = -Q^{(1)}(\mathbf{r}). \tag{1}$$

The solution $p^{(1)}(\mathbf{r})$ satisfies the radiation condition at infinitely distant points of the space Ω and the boundary conditions

$$\left.\frac{\partial p^{(1)}(\mathbf{r})}{\partial n}\right|_{S_i} = -\rho\left(-i\omega + V\frac{\partial}{\partial x}\right)^2 w_i^{(1)}(\mathbf{r}) \equiv -M w_i^{(1)}(\mathbf{r}); \tag{2}$$

$$L_i w_i^{(1)}(\mathbf{r}) = F_i^{(1)}(\mathbf{r}) - \left.p^{(1)}(\mathbf{r})\right|_{S_i}; \tag{3}$$

$$T_{ij}\left(w_i^{(1)}\right) = g_{ij}^{(1)}\big|_{\Gamma'}, \qquad R_{ij}\left(w_i^{(1)}\right) = -f_{ij}^{(1)}\big|_{\Gamma''}, \qquad \Gamma_i' + \Gamma_i'' = \Gamma_i. \tag{4}$$

Assume that there is another system of sources $Q^{(2)}(\mathbf{r})$ and that the direction of the flow in Ω is opposite to its direction in the first case. The field $\tilde{p}^{(2)}(\mathbf{r})$ produced by the sources $Q^{(2)}(\mathbf{r})$ obeys the equation

$$\Delta\tilde{p}^{(2)}(\mathbf{r}) - \frac{1}{c^2}\left(-i\omega - V\frac{\partial}{\partial x}\right)^2 \tilde{p}^{(2)}(\mathbf{r}) = -Q^{(2)}(\mathbf{r}), \tag{5}$$

adjoint to (1). The solution $\tilde{p}^{(2)}(\mathbf{r})$ satisfies the radiation condition, the boundary condition

$$\left.\frac{\partial\tilde{p}^{(2)}(\mathbf{r})}{\partial n}\right|_{S_i} = -\rho\left(-i\omega - V\frac{\partial}{\partial x}\right)^2 \widetilde{w}_i^{(2)}(\mathbf{r}) \equiv -\widetilde{M}\,\widetilde{w}_i^{(2)}(\mathbf{r}) \tag{6}$$

and conditions (3) and (4), where the superscript (1), as in equations (3) and (6), is replaced by the superscript (2).

In expressions (2)–(6): $\mathbf{n}$ is the normal to S_i; $w_i(\mathbf{r})$ are the normal displacements of the surface of the shell; $F_i(\mathbf{r})$ are external mechanical harmonic forces acting on the shell in the direction of the normal; g_{ij}, f_{ij} are external forces, moments, displacements, etc., acting on the shell along its contour Γ_i (at the edges of the shell); L_i is a self-adjoint differential operator, for which Green' s formula is valid,

$$\int_{S_i}\left[w_i^{(1)}(\mathbf{r})L_i\widetilde{w}_i^{(2)}(\mathbf{r}) - \widetilde{w}_i^{(2)}(\mathbf{r})L_i w_i^{(1)}(\mathbf{r})\right]\,ds(\mathbf{r}) =$$

$$= \int_{\Gamma_i}\sum_{j=1}^{N}\left[R_{ij}(\widetilde{w}^{(2)}(\mathbf{r}))\,T_{ij}(w^{(1)}(\mathbf{r})) - T_{ij}(\widetilde{w}_i^{(2)}(\mathbf{r}))\,R_{ij}(w^{(1)}(\mathbf{r}))\right]\,dl(\mathbf{r}); \tag{7}$$

M and $\widetilde{M}$ are adjoint differential operators that commute with the operator L_i; R_{ij} and T_{ij} are also certain differential operators, and N depends on the order of the differential operator L_i.*

Multiply equation (1) by $\tilde{p}^{(2)}(\mathbf{r})$, and equation (5) by $-p^{(1)}(\mathbf{r})$, add them, and integrate both sides of the resulting equality over the volume Ω. Applying Green' s formula to the volume integral on the left-hand side of the equality, we write:

$$\int_{\Omega} Q^{(1)}(\mathbf{r})\tilde{p}^{(2)}(\mathbf{r})\,d\Omega(\mathbf{r}) + \sum_{i=1}^{k}\int_{S_i}\frac{\partial}{\partial n}\tilde{p}^{(2)}(\mathbf{r})\,p^{(1)}(\mathbf{r})\,ds(\mathbf{r}) =$$

$$= \int_{\Omega} Q^{(2)}(\mathbf{r}) p^{(1)}(\mathbf{r})\, d\Omega(\mathbf{r}) + \sum_{i=1}^{k} \int_{S_i} \frac{\partial}{\partial n} p^{(1)}(\mathbf{r})\, \tilde{p}^{(2)}(\mathbf{r})\, ds(\mathbf{r}). \tag{8}$$

Using the boundary conditions for equations (1) and (5) and formula (7), we finally obtain

$$\int_{\Omega} Q^{(2)}(\mathbf{r}) p^{(1)}(\mathbf{r})\, d\Omega(\mathbf{r}) + \sum_{i=1}^{k} \int_{S_i} \frac{\partial p^{(1)}(\mathbf{r})}{\partial n} F_i^{(2)}(\mathbf{r})\, ds(\mathbf{r}) +$$

$$+ \sum_{i=1}^{n} \int_{\Gamma_i} \sum_{j=1}^{N} R_{ij} \left(\frac{\partial p^{(1)}(\mathbf{r})}{\partial n} \right) g_{ij}^{(2)}(\mathbf{r})\, dl(\mathbf{r}) +$$

$$+ \sum_{i=1}^{n} \int_{\Gamma_i} \sum_{j=1}^{N} T_{ij} \left(\frac{\partial p^{(1)}(\mathbf{r})}{\partial n} \right) f_{ij}^{(2)}(\mathbf{r})\, dl(\mathbf{r}) = \int_{\Omega} Q^{(1)}(\mathbf{r}) \tilde{p}^{(2)}(\mathbf{r})\, d\Omega(\mathbf{r}) +$$

$$+ \sum_{i=1}^{k} \int_{S_i} \frac{\partial \tilde{p}^{(2)}(\mathbf{r})}{\partial n} F_i^{(1)}(\mathbf{r})\, ds(\mathbf{r}) + \sum_{i=1}^{m} \int_{\Gamma_i} \sum_{j=1}^{N} R_{ij} \left(\frac{\partial \tilde{p}^{(2)}(\mathbf{r})}{\partial n} \right) g_{ij}^{(1)}(\mathbf{r})\, dl(\mathbf{r}) +$$

$$+ \sum_{i=1}^{m} \int_{\Gamma_i} \sum_{j=1}^{N} T_{ij} \left(\frac{\partial \tilde{p}^{(2)}(\mathbf{r})}{\partial n} \right) f_{ij}^{(1)}(\mathbf{r})\, dl(\mathbf{r}). \tag{9}$$

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* We note that in the case of thin homogeneous rods, plates, and cylindrical shells of circular cross section, the operators M and $\widetilde{M}$ commute with the operator L_i, since L_i is a differential operator with constant coefficients. In other cases, the existence of commutativity must be proved in advance.

Expression (9) may be regarded as an integral relation of the reciprocity type in the acoustics of a moving medium.

Let us consider special cases. If $Q^{(1)}(\mathbf{r}) = Q^{(2)}(\mathbf{r}) = 0$, $F_i^{(1)}(\mathbf{r}) = F_i^{(2)}(\mathbf{r}) = 0$, $i = 2, 3, \dots, k$, then from expression (8) it follows that

$$\int_{S} \frac{\partial p^{(1)}(\mathbf{r})}{\partial n} \tilde{p}^{(2)}(\mathbf{r})\, ds(\mathbf{r}) = \int_{S} \frac{\partial \tilde{p}^{(2)}(\mathbf{r})}{\partial n} p^{(1)}(\mathbf{r})\, ds(\mathbf{r}), \tag{10}$$

and from (9) we obtain

$$\int_{S} \frac{\partial p^{(1)}(\mathbf{r})}{\partial n} F^{(2)}(\mathbf{r})\, ds(\mathbf{r}) = \int_{S} \frac{\partial \tilde{p}^{(2)}(\mathbf{r})}{\partial n} F^{(1)}(\mathbf{r})\, ds(\mathbf{r}). \tag{11}$$

The integral relation (10) is known as the reciprocity relation and the flow-reversal theorem in the aerodynamics of a compressible gas (³). This theorem is widely used for solving problems of aeroelasticity (⁴). Formula (11) in essence follows from the integral relation (10) and is valid if the operators M and $\widetilde{M}$ commute with the operator L_i.

Suppose that

$$Q^{(1)}(\mathbf{r}) = F_i^{(2)}(\mathbf{r}) = g_{ij}^{(1)}(\mathbf{r}) = g_{ij}^{(2)}(\mathbf{r}) = f_{ij}^{(1)}(\mathbf{r}) = f_{ij}^{(2)}(\mathbf{r}) = 0, \quad i = 1, 2, \dots, k;$$

$$F_i^{(1)}(\mathbf{r}) = 0, \quad i = 2, 3, \dots, k.$$

Then we obtain

$$\int_\Omega Q^{(2)}(\mathbf{r})\, p^{(1)}(\mathbf{r})\, d\Omega(\mathbf{r}) = \int_S \frac{\partial \tilde{p}^{(2)}(\mathbf{r})}{\partial n}\, F^{(1)}(\mathbf{r})\, ds(\mathbf{r}). \tag{12}$$

Relation (12) connects the solutions of boundary-value problems of sound diffraction by thin elastic bodies situated in a gas flow and of sound radiation by elastic bodies in a flow, oscillating under the action of external forces.

Let

$$F_i^{(1)}(\mathbf{r}) = F_i^{(2)}(\mathbf{r}) = 0, \quad i = 1, 2, \dots, k;$$

$$g_{ij}^{(1)}(\mathbf{r}) = g_{ij}^{(2)}(\mathbf{r}) = f_{ij}^{(1)}(\mathbf{r}) = f_{ij}^{(2)}(\mathbf{r}) = 0, \quad i = 1, 2, \dots, k.$$

In this case, on the basis of expression (9), one may write

$$\int_\Omega Q^{(2)}(\mathbf{r})\, p^{(1)}(\mathbf{r})\, d\Omega(\mathbf{r}) = \int_\Omega Q^{(1)}(\mathbf{r})\, \tilde{p}^{(2)}(\mathbf{r})\, d\Omega(\mathbf{r}). \tag{13}$$

Formula (13) establishes a connection between the solutions of two adjoint boundary-value problems of sound diffraction by elastic bodies in a moving medium. It is important to note that relation (13) is valid when, in the space occupied by the flow, there are no boundaries, or there are perfectly soft and rigid boundaries, as is evident from consideration of formula (8), and also in the presence of elastic boundaries, when the commutativity conditions for the operators M, $\widetilde{M}$, and L_i are satisfied. If the commutativity conditions are not satisfied, expression (13) is not valid and must be replaced by the integral relation (8).

As above, on the basis of formula (9), integral formulas may be obtained that connect the solutions of problems of diffraction and radiation of sound by shells in a moving medium, when the shell undergoes oscillations under the action of prescribed external harmonic forces, moments, displacements, etc., at its edges (contour Γ_i.

Let us note some possibilities for applying the integral relations obtained. In particular, it follows from them that the solution of the problem of sound radiation by shells undergoing oscillations under the action of prescribed forces, displacements, moments, regularly or statistically distributed over the surface or edges of the shell, is immediately reduced to the computation of qua-

···equipment, if the solution of the corresponding homogeneous diffraction problem is known. Indeed, suppose it is necessary to determine the radiation field $p^{(1)}(\mathbf{r})$ at some point $\mathbf{r}_1$ of the space Ω. Placing at this point an auxiliary point source $Q_0^2\delta(\mathbf{r}-\mathbf{r}_1)$, whose field $\tilde{p}^{(2)}(\mathbf{r})$ is known, we obtain, on the basis of (12),

$$p^{(1)}(\mathbf{r}_1) = \frac{1}{Q_0^{(2)}} \int_S \frac{\partial \tilde{p}^{(2)}(\mathbf{r})}{\partial n} F^{(1)}(\mathbf{r})\, ds(\mathbf{r}). \tag{14}$$

In the case of statistically distributed forces we write

$$\overline{\left|p^{(1)}(\mathbf{r}_1)\right|^2} = \frac{1}{\left|Q_0^{(2)}\right|^2} \iint_{SS} \frac{\partial \tilde{p}^{(2)}(\mathbf{r}')}{\partial n} \frac{\partial p^{(2)*}(\mathbf{r}'')}{\partial n} \overline{F^{(1)}(\mathbf{r}') F^{(1)*}(\mathbf{r}'')}\, ds(\mathbf{r}')\, ds(\mathbf{r}''). \tag{15}$$

Analogous formulas can be obtained in other cases.

If one of the solutions, for example $\tilde{p}^2(\mathbf{r})$, is regarded as the source function for the adjoint equation with homogeneous boundary conditions $F_i^{(2)}(\mathbf{r}) = f_i^{(2)}(\mathbf{r}) = g_i^{(2)}(\mathbf{r}) = 0$, then the integral relation (9) represents a generalized Green formula for the acoustics of a moving medium.

In conclusion, we note that relation (9) can be generalized to the case of non-stationary oscillations. Other generalizations are also possible.

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Note: Figure translations are in progress. See original paper for figures.