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Soviet-era science, translated into English

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1961

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **REGULAR AND PROJECTION-COMPLETE LINEALS IN SPACES WITH A GENERAL HERMITIAN-BILINEAR METRIC**

*(Presented by Academician L. S. Pontryagin on 21 III 1961)*

1. Consider an infinite-dimensional complex linear system (space)  $\mathfrak{C}$ , in which a Hermitian bilinear functional  $G(x, y) = [x, y]$ , or, briefly, a  $G$ -metric, is given. Since this  $G$ -metric is, generally speaking, indefinite, vectors  $x \in \mathfrak{C}$  may be positive, null, and negative ( $[x, x] \geq 0$ , respectively,  $x \neq 0$ ). A lineal  $\mathfrak{L} (\subseteq \mathfrak{C})$  will be called positive (negative) if all nonzero vectors  $x \in \mathfrak{L}$  are positive (negative). Positive and negative lineals are called definite.

Let  $\mathfrak{L}$  be some lineal in  $\mathfrak{C}$ , and let  $y_0 \in \mathfrak{C}$ . A vector  $x_0 \in \mathfrak{L}$  is called the  $G$ -projection of the vector  $y_0$  onto  $\mathfrak{L}$ , if  $[x, y_0 - x_0] = 0$  for all  $x \in \mathfrak{L}$  (i.e. the vector  $y_0 - x_0$  is “ $G$ -orthogonal” to  $\mathfrak{L}$ ).

The question of the existence of  $G$ -projections was studied in the literature <sup>(1-3)</sup> for the special case in which  $\mathfrak{C} = \mathfrak{H}$  is a Hilbert space with scalar product  $(x, y)$  ( $x, y \in \mathfrak{H}$ ), and the  $G$ -metric is given by means of the form

$$[x, y] = (Gx, y) \quad (x, y \in \mathfrak{H}),$$

where  $G$  is a bounded Hermitian\* operator in  $\mathfrak{H}$  (the so-called Gram operator of the form  $[x, y]$  in  $\mathfrak{H}$  <sup>(3)</sup>). Still earlier (and, apparently, for the first time)  $G$ -projections were studied by L. S. Pontryagin <sup>(4)</sup> for the case in which  $G = P^+ - P^-$ , where  $P^+$ ,  $P^-$  are orthoprojectors in  $\mathfrak{H}$ ,  $P^+P^- = P^-P^+ = 0$ ,  $P^+ + P^- = I$ , and  $\min(\dim P^+\mathfrak{H}, \dim P^-\mathfrak{H}) = \varkappa < \infty$ . In papers <sup>(5, 6)</sup> a space with such a  $G$ -metric was denoted by  $\Pi_\varkappa$ . We shall adhere to this notation here as well.

In all the papers mentioned, the natural requirement of closedness in the norm

$$\|x\| = (x, x)^{1/2} \quad (x \in \mathfrak{H}),$$

was imposed on the lineal  $\mathfrak{L}$ , i.e.  $\mathfrak{L}$  was a subspace of the Hilbert space  $\mathfrak{H}$ . With the aid of the so-called canonical decomposition of the subspace  $\mathfrak{L}$  <sup>(1, 2)</sup>, the question of  $G$ -projection onto an arbitrary subspace was reduced to  $G$ -projection onto definite subspaces.

In the general case, i.e. for an arbitrary linear system  $\mathfrak{E}$ , not endowed with any topology, the methods used in (1-4) become inapplicable, even if one restricts  $G$ -projection only to definite lineals. Meanwhile, such a problem has recently arisen more than once in a number of works on quantum field theory (see, for example, (7, 8)). The present note is devoted to the solution of this problem and to some related questions.

\* In paper (3),  $G$  is an arbitrary bounded operator.

2. Let  $\mathfrak{L} (\subset \mathfrak{E})$  be a definite lineal, and let  $y$  be some vector of  $\mathfrak{E}$ . Consider the linear form (functional)  $f_y(x) = [x, y]$ . If in  $\mathfrak{L}$  one introduces the norm

$$|x| = |[x, x]|^{1/2} \quad (x \in \mathfrak{L}),$$

then, with respect to it, the functional  $f_y(x)$  may turn out to be continuous (bounded) or discontinuous. If, for any vector  $y \in \mathfrak{E}$ , the functional  $f_y(x)$  is continuous with respect to  $|x|$  ( $x \in \mathfrak{L}$ ), then the lineal  $\mathfrak{L}$  will be called **regular**. If, however, for some  $y_0 \in \mathfrak{E}$  the functional  $f_{y_0}(x)$  is discontinuous with respect to  $|x|$ , then the lineal  $\mathfrak{L}$  will be called **singular**.

**Theorem 1.** Let  $\mathfrak{L} (\subset \mathfrak{E})$  be a definite lineal, and let  $y_0 \in \mathfrak{E}$ . In order that the functional  $f_{y_0}(x) = [x, y_0]$  be bounded with respect to the norm  $|x|$ , ( $x \in \mathfrak{L}$ ), it is necessary and sufficient that

$$\inf_{x \in \mathfrak{L}} [x - y_0, x - y_0] = m(y_0) > -\infty, \quad (1)$$

if the lineal  $\mathfrak{L}$  is positive, and

$$\sup_{x \in \mathfrak{L}} [x - y_0, x - y_0] = M(y_0) < +\infty, \quad (1')$$

if the lineal  $\mathfrak{L}$  is negative.

Thus, a definite lineal  $\mathfrak{L}$  is regular if and only if condition (1) (respectively (1')) is fulfilled for all  $y_0 \in \mathfrak{E}$ .

We shall formulate another criterion for the regularity of a definite lineal in terms of **singular** lineals. We shall call a  $G$ -metric  $[x, y]$  **nondegenerate on the lineal  $\mathfrak{L}$**  if from the equality  $[x_0, y] = 0$ , valid for some  $x_0 \in \mathfrak{L}$  and all  $y \in \mathfrak{L}$ , it follows that  $x_0 = 0$ . In the opposite case ( $x_0 \neq 0$ ) one says that the  $G$ -metric **degenerates on  $\mathfrak{L}$** , and  $x_0$  is called an **isotropic vector of the lineal  $\mathfrak{L}$**  (5).

**Theorem 2.** Let  $\mathfrak{L}$  be a definite lineal in the space  $\mathfrak{E}$  with a nondegenerate  $G$ -metric. In order that the lineal  $\mathfrak{L}$  be singular, it is necessary and sufficient that  $\mathfrak{L} \subset \mathfrak{E}_1 \subset \mathfrak{E}$ , where  $\mathfrak{E}_1$  is a linear system admitting a completion (see (5), Theorem 1.4) to the space  $\Pi_1$ , after which the closure  $\bar{\mathfrak{L}}$  of the lineal  $\mathfrak{L}$  in the norm of the space  $\Pi_1$  contains vectors isotropic (in  $\mathfrak{L}$ ).

In the work <sup>(8)</sup>, A. Wightman proceeds from the assumption that in the case of a **complete** definite lineal  $\mathfrak{L}$ , with respect to the norm  $|x| = |[x, x]|^{1/2}$  ( $x \in \mathfrak{L}$ ), this lineal is regular. This assertion was used by A. Wightman as the basis of the scheme of quantum mechanics in a space with an indefinite metric developed by him.\* Meanwhile, it is easy to give examples refuting A. Wightman's hypothesis. In essence, such an example of a singular lineal complete in the norm  $|x|$  had already been constructed in <sup>(6)</sup> on p. 421; moreover, as the following theorem shows, this example exhausts the entire class of similar lineals.

**Theorem 3.** In order that a singular lineal  $\mathfrak{L} (\subset \mathfrak{E})$  be complete with respect to the norm  $|x| = |[x, x]|^{1/2}$ , it is necessary and sufficient that, after the isometric embedding of  $\mathfrak{L}$  into the space  $\Pi_1$  (according to Theorem 2), its closure  $\bar{\mathfrak{L}}$  admit a decomposition into the direct sum  $\bar{\mathfrak{L}} = \mathfrak{L} \dot{+} \{x_0\}$ , where  $\{x_0\}$  is a one-dimensional subspace spanned by an isotropic vector  $x_0$  of the subspace  $\bar{\mathfrak{L}}$ .

3. The question of the regularity of a definite lineal is closely connected with the problem of  $G$ -projection. We shall call a definite lineal  $\mathfrak{L} (\subset \mathfrak{E})$  **projection-complete** if, for every vector  $y_0 \in \mathfrak{E}$ , there exists a  $G$ -projection  $x_0$  onto the lineal  $\mathfrak{L}$ . It is obvious that a **necessary** condition for projection-completeness is the regularity of the lineal. The following

\* Our attention was drawn to work <sup>(8)</sup> by the Hungarian mathematician János Bognár, who expressed the first considerations calling A. Wightman's hypothesis into question and gave examples of (incomplete) singular lineals in  $\mathfrak{E}$ .

Theorem 4 establishes a criterion for the projection completeness of a definite lineal.

**Theorem 4.** In order that a vector  $y_0 \in \mathfrak{E}$  have a  $G$ -projection  $x_0$  onto a definite lineal  $\mathfrak{L} \subset \mathfrak{E}$ , it is necessary and sufficient that the following two conditions be satisfied:

- 1°. The functional  $f_{y_0}(x) = [x, y_0]$  ( $x \in \mathfrak{L}$ ) is bounded with respect to the norm  $|x|$ . We denote the norm of this functional by  $|f_{y_0}|$ .
- 2°. For at least one sequence  $\{x_n\} \subset \mathfrak{L}$  having the property

$$\lim_{n \rightarrow \infty} \frac{[x_n, y_0]}{|x_n|} = |f_{y_0}|, \quad (2)$$

and normalized by the conditions  $|x_n| = |f_{y_0}|$  ( $n = 1, 2, \dots$ ), there exists a limit  $x_0 \in \mathfrak{L}$ :  $\lim_{n \rightarrow \infty} |x_n - x_0| = 0$ . The vector  $\{\text{sign}[x_0, x_0]\} \cdot x_0$  is then the  $G$ -projection of the vector  $y_0$  onto the lineal  $\mathfrak{L}$ .

Thus, the lineal  $\mathfrak{L}$  will be projection complete if and only if it is regular and condition 2° is satisfied for all vectors  $y_0 \in \mathfrak{E}$ .

Let us note that if condition 2° is satisfied for a given  $y_0$  for at least one normalized sequence  $\{x_n\}$  having property (2), then it will also be satisfied for **any** such sequence, and the vector  $x_0$  is determined uniquely. It is easy to see that

any sequence having property (2) is fundamental with respect to  $|x|$  (cf. (1)). Therefore condition 2° will in any case be satisfied for all vectors  $y_0 \in \mathfrak{E}$ , if the lineal  $\mathfrak{L}$  is **complete** with respect to the norm  $|x|$ .

Thus, *regularity and completeness of a definite lineal  $\mathfrak{L}$  entail its projection completeness*. At the same time, by Theorems 3 and 4, completeness of the lineal  $\mathfrak{L}$  alone is not sufficient for its projection completeness.\*

4. Returning to Theorem 1, we point out that condition (1) (respectively, (1')) was first established by R. Nevanlinna<sup>(1)</sup> as **necessary** for the existence of the  $G$ -projection of a vector  $y_0$  onto a definite lineal  $\mathfrak{L}$  under the **additional restrictions**:

- 1)  $\mathfrak{E} = \mathfrak{H}$  is a Hilbert space with  $G$ -metric (see item 1).
- 2)  $\mathfrak{L}$  is a (closed) subspace of  $\mathfrak{H}$ .

There it is also shown<sup>(1)</sup> that condition (1) (respectively, (1')) is **sufficient** if  $\mathfrak{L}$  is complete (with respect to the norm  $|x|$ ).

Comparison of Theorems 1 and 4 formulated by us shows that the restrictions 1) and 2) imposed by R. Nevanlinna are superfluous.

In the work<sup>(3)</sup> of Yu. P. Ginzburg, under the same restrictions 1) and 2), there was first obtained a **necessary and sufficient** condition for the existence of the  $G$ -projection of a vector  $y_0 \in \mathfrak{H}$  onto a subspace  $\mathfrak{L}$  (even for the case of a non-Hermitian operator  $G$ ). If  $P_{\mathfrak{L}}$  is the orthoprojector from  $\mathfrak{H}$  onto  $\mathfrak{L}$ , then  $G_{\mathfrak{L}} = P_{\mathfrak{L}}GP_{\mathfrak{L}}$  is the Gram operator of the form  $[x, y]$  on the subspace  $\mathfrak{L}$  (see item 1). It turns out<sup>(3)</sup>, Theorem 2) that in the case of a Hermitian operator  $G$  of interest to us, *for the existence of the  $G$ -projection of a vector  $y_0$  onto  $\mathfrak{L}$ , it is necessary and sufficient that for every sequence  $\{x_n\} \subset \mathfrak{L}$  having the property*

$$\lim_{n \rightarrow \infty} \|G_{\mathfrak{L}}x_n\| = 0,$$

*the equality*

$$\lim_{n \rightarrow \infty} [x_n, y_0] = 0$$

*hold.*

\* Already after submitting the present article for publication, we became acquainted with the work<sup>(9)</sup>, in wh

The necessity of this condition is directly evident from the relation

$$[x, y_0] = [x, x_0] = (G_{\Omega}x, x_0) \quad (x \in \Omega),$$

in which  $x_0$  denotes the  $G$ -projection of the vector  $y_0$  onto  $\Omega$ . Using the closedness of the operator  $G_\Omega^{-1}$  with respect to the norm  $|x|$ , it is not difficult to show that, in its sufficient part, the criterion under consideration follows from our Theorem 4.

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Received  
17 III 1961

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*Note: Figure translations are in progress. See original paper for figures.*

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