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Abstract

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ON MULTIPLICATIVE SEMIGROUPS OF RINGS

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At the Third All-Union Colloquium on General Algebra, A. I. Mal' cev put forward the hypothesis that the class of multiplicative semigroups of rings is not axiomatizable. In the present note the validity of this hypothesis is proved. Moreover, the nonaxiomatizability of the class of multiplicative semigroups of fields is proved, and consequently of every class of groupoids within which the latter is axiomatizable. The exposition of the main content of the note is preceded by a theorem on axiomatizable classes.

§ 1. Let V be a nonempty class, and Ω some set of its subclasses, which is a Boolean algebra. We shall say that an element b is Ω-inseparable from an element a ($a, b \in \Omega$), if in Ω there are no classes containing a and not containing b. A class $K \subset V$ will be called weakly Ω-closed if, for every element a, from the existence in K of an element Ω-inseparable from a, it follows that $a \in K$. It is easy to see that, for every element b, the Ω-closure $\{b\}$ (see ([3])) contains all and only those elements from which b is Ω-inseparable. Hence it follows that all possible unions of Ω-closed classes, and only they, are weakly Ω-closed. We shall call the operation of Ω-closure compact if the family of Ω-closed classes is compact in the sense of ([4]), or, what is the same, if every chain of nonempty Ω-closed classes has a nonempty intersection.

Lemma. *Let the class K be such that the closure operation induced on K by the Ω-closure operation is compact. Then the weak Ω-closedness of K is equivalent to its Ω-closedness.*

From the characterization of weakly Ω-closed classes given above it follows that an Ω-closed class is weakly Ω-closed. Let the class K be weakly Ω-closed, and let a be a point of contact of K with respect to Ω. The latter means that all classes from Ω containing a intersect K. Since Ω is a lower semilattice, all possible finite intersections of classes from Ω containing a intersect K. Since the closure operation induced on K by the Ω-closure operation is compact, it follows that all classes from Ω containing a have a common element in K. The latter is Ω-inseparable from a, and therefore $a \in K$.

Let V be the class of all models of an arbitrary fixed type. We shall assume that the models under consideration belong to V, and that the axioms are sentences of the elementary formal theory corresponding to V.

Let a class of models K be such that, if every finite subsystem of an arbitrary given system of axioms is satisfied in a suitable model from K, then the whole system is satisfied in a suitable model from K (cf. ([5])). Then we shall call it a **pseudoaxiomatizable clas-**

s.o.m. From the Gödel–Mal' cev theorem it follows that every PC_Δ-class (see ([2])) is pseudoaxiomatizable.

If AC is the set of all finitely axiomatizable classes of models, then AC-closedness of a class means its axiomatizability, weak AC-closedness means arithmetic closedness (see ([1])); pseudoaxiomatizability of a class is equivalent to saying that the closure operation induced on it by the operation of AC-closure is compact. Thus we have:

Theorem 1. *A class of models is axiomatizable if and only if it is pseudoaxiomatizable and arithmetically closed* (cf. ([7])).

§ 2. Following ([6]), we shall call an abelian group a **group of the first kind** if the orders of its elements are bounded in the aggregate; otherwise, a **group of the second kind**. If $\mathfrak{G}$ is a group of the second kind and $\mathfrak{R}$ is the additive group of the rational numbers, then $\mathfrak{G} \times \mathfrak{R}$ and $\mathfrak{G}$ are arithmetically equivalent (see ([6]), Theorem 6.6).

Let $\mathfrak{G}$ be the multiplicative group of the algebraic closure of the field with two elements. It is a group of the second kind and therefore is arithmetically equivalent to $\mathfrak{G} \times \mathfrak{R}$.

We shall show that $\mathfrak{G} \times \mathfrak{R}$ is not the multiplicative group of any field.

Suppose that $\mathfrak{G} \times \mathfrak{R}$ is the multiplicative group of some field $\mathfrak{K}$. The characteristic of $\mathfrak{K}$ must be equal to 2, since in this field the equation $x^2 = 1$ has multiple roots. Every algebraic element over the prime subfield of $\mathfrak{K}$ is a root of unity. Therefore all elements $\langle a, r\rangle \in \mathfrak{G} \times \mathfrak{R}$, where $r \neq 0$, are transcendental over the prime subfield of the field $\mathfrak{K}$ (since they have infinite order in $\mathfrak{G} \times \mathfrak{R}$). From the existence in $\mathfrak{K}$ of elements transcendental over the prime subfield there follows the existence in $\mathfrak{G} \times \mathfrak{R}$ of an infinite linearly independent system of elements. On the other hand, any two elements of $\mathfrak{G} \times \mathfrak{R}$ are linearly dependent.

Indeed, let $\langle a_1, r_1\rangle$ and $\langle a_2, r_2\rangle$ be arbitrary elements of $\mathfrak{G} \times \mathfrak{R}$; let n_1 and n_2 be natural numbers such that $n_1 r_1 + n_2 r_2 = 0$. There exists an m such that $(a_1^{n_1} \cdot a_2^{n_2})^m = e$, where e is the identity of the field $\mathfrak{K}$. Then

$$mn_1\langle a_1, r_1\rangle + mn_2\langle a_2, r_2\rangle = \langle a_1^{mn_1} \cdot a_2^{mn_2},\, m(n_1 r_1 + n_2 r_2)\rangle = \langle e, 0\rangle$$

is the zero of the group $\mathfrak{G} \times \mathfrak{R}$ (we used the additive notation for the group $\mathfrak{G} \times \mathfrak{R}$).

The contradiction obtained proves that $\mathfrak{G} \times \mathfrak{R}$ is not the multiplicative group of any field. Thus we have:

Theorem 2. *The class of multiplicative groups of fields is not arithmetically closed, and therefore is not axiomatizable.*

It follows from this that every class of groupoids within which the class of multiplicative groups of fields is axiomatizable is non-axiomatizable; in particular, so is the class of multiplicative groups of division rings.

From the arguments given above proving Theorem 2 it follows that the class of multiplicative groups of fields of characteristic 2 is not arithmetically closed. In exactly the same way one can show that the class of multiplicative groups of fields of any finite characteristic is not arithmetically closed. For fields of characteristic zero the question remains open.

Let $\mathfrak{G}$ denote some group, written multiplicatively. Then by $\mathfrak{G}^*$ we shall denote the semigroup obtained by adjoining to $\mathfrak{G}$ an external zero (i.e. such an element ω that $\omega a = a\omega = \omega$ for every $a \in \mathfrak{G}$). It is easy to verify that if the groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are arithmetically equivalent, then $\mathfrak{G}_1^*$ and $\mathfrak{G}_2^*$ are arithmetically equivalent. From this fact and Theorem 2 it follows:

Theorem 3. *The class of multiplicative semigroups of fields is not axiomatizable. Consequently, every class of groupoids within which the class of multiplicative semigroups of fields is axiomatizable is not axiomatizable.*

In particular, the class of multiplicative semigroups of associative rings and the class of multiplicative groupoids of all, in general nonassociative, rings are not axiomatizable.

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Note: Figure translations are in progress. See original paper for figures.