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A. P. SAVIN

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Abstract

Full Text

MATHEMATICS

A. P. SAVIN

ON DECOMPOSITIONS OF COMPLETELY NORMAL SPACES

(Presented by Academician P. S. Aleksandrov on 16 II 1961)

In the present paper the concept of a decomposition of a given space X by its closed subset A is generalized, and an equivalence of generalized decompositions is introduced. On the set of the resulting equivalence classes a group operation is introduced. It is proved that the groups thus constructed are naturally isomorphic to the cohomotopy groups of the space X.

1. Decompositions of order k. Let X be a topological space. An **elementary decomposition** of the space X will mean an ordered triple of subsets of this space

$$\alpha = (A, U, V),$$

consisting of a closed set A and open sets U and V, possessing the property that

$$U \cap V = \emptyset, \qquad U \cup V = X \setminus A.$$

An ordered pair $\sigma^0 = (B, C)$ of closed subsets of some space Y will be called a **decomposition of order zero** of the space Y, if

$$B \cap C = \emptyset, \qquad B \cup C = Y.$$

Suppose k elementary decompositions

$$\alpha_i = (A_i, U_i, V_i), \quad i = 1, 2, \dots, k,$$

of the space X are given, and a decomposition (B, C) of order zero of the set

$$\bigcap_{i=1}^{k} A_i.$$

In this case the ordered collection

$$\sigma^k = (\alpha_1, \dots, \alpha_k, B, C)$$

will be called a **decomposition of order** k of the space X.

2. Equivalence classes. Two decompositions of order k of the space X will be called **equivalent** if there exists a decomposition of order k of the space $X \times I$,

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where I is the unit interval $[0, 1]$, such that on the upper base $X \times 1$ it coincides with one of the given decompositions, and on the lower base $X \times 0$ with the other. The concept of equivalence introduced is, evidently, reflexive, symmetric, and transitive. The set of all equivalence classes of decompositions of order k of the space X will be denoted by $P^k(X)$.

3. Functions and decompositions. In the present paper the space X is assumed to be completely normal, i.e. to have the property that every one of its closed subsets is of type G_δ. The basic properties of a completely normal space used in the present paper are the following:

Property 3.1. Let A and B be closed sets of the space X and

$$A \cap B = \emptyset;$$

then there exists on the space X a real continuous function $f(x)$ satisfying the conditions:

$$f(x) = 0, \quad \text{if } x \in A; \qquad f(x) = 1, \quad \text{if } x \in B; \qquad 0 < f(x) < 1, \quad \text{if } x \in X \setminus (A \cup B).$$

Property 3.2. In order that the space X be completely normal, it is necessary and sufficient that every mapping of an arbitrary closed set $A \subset X$ into the sphere S^{k-1}, which is the boundary of the ball T^k, can be extended to a mapping

$$F : X \to T^k,$$

having the property that

$$F^{-1}(S^{k-1}) = A.$$

Proof. Sufficiency follows from the fact that the preimage under a continuous mapping F of a set of type G_δ is in turn a set of type G_δ in the mapped space.

We prove necessity. Let $F_1 : X \to T^k$ be some extension of the mapping $f : A \to S^{k-1}$ (such an extension exists, since the space X is, obviously, normal). Further, let $\varphi(x)$ $(0 \le \varphi(x) \le 1)$ be a real continuous function on the space X possessing the property that $\varphi(x) = 0$ if and only if $x \in A$. Let O be the center of the ball T^k, whose radius we shall assume to be equal to one. Denote by Γ_λ the homothety of the ball T^k with center O and coefficient λ. Then the mapping $F : K \to T^k$, defined by the formula $F(x) = \Gamma_{1-\varphi(x)}(F_1)(x)$, $x \in X$, is, obviously, continuous and satisfies the condition.

Let $f(x)$ be a continuous function on the space X; consider the subsets $A_f = (x \mid f(x) = 0)$, $U_f = (x \mid f(x) > 0)$, $V_f = (x \mid f(x) < 0)$ of the space X. As is easy to see, the triple $\alpha_f = (A_f, U_f, V_f)$ is an elementary decomposition of the space X, which we shall call the elementary decomposition of the space X generated by the function f.

4. Properties of decompositions. If a certain collection of decompositions $\{\sigma_z^k\}$ of order k of the space X is given, where z is some index corresponding to

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the fixed decomposition, then we shall agree to write the decomposition σ_z^k in the form $(\alpha_1^z, \dots, \alpha_k^z, B^z, C^z)$, where $\alpha_i^z = (A_i^z, U_i^z, V_i^z)$.

Property 4.1. Let σ_1^k and σ_2^k be two decompositions of order k of the space X, and let $A_i^1 \subset A_i^2$, $U_i^1 \supset U_i^2$, $V_i^1 \supset V_i^2$, $i = 1, 2, \dots, k$, $B^1 \subset B^2$, $C^1 \subset C^2$; then the decompositions σ_1^k and σ_2^k are equivalent.

Indeed, consider in the space $X \times I$ the closed sets: $A_i = (A_i^1 \times I) \cup A_i^2$, $B = (B^1 \times I) \cup B^2$, $C = (C^1 \times I) \cup C^2$, and the open sets: $U_i = (U_i^1 \times I) \setminus A_i$, $V_i = (V_i^1 \times I) \setminus A_i$. It is easy to see that $\alpha_i = (A_i, U_i, V_i)$ are elementary decompositions of the space $X \times I$, while $\sigma^k = (\alpha_1, \dots, \alpha_k, B, C)$ is a decomposition of order k of the space $X \times I$, establishing the equivalence of the decompositions σ_1^k and σ_2^k.

Property 4.2. Let a decomposition σ_1^k of order k of the space X be given, and let a neighborhood $O(B^1)$ of the set B^1 be given; then there exists a decomposition $\sigma^k \sim \sigma_1^k$ such that

$$\bigcup_{i=1}^{k} (A_i \cup V_i) \subset O(B^1).$$

Indeed, let $W(B^1)$ be a neighborhood of the set B^1 lying together with its closure $[W(B^1)]$ in $O(B^1)$ (such a neighborhood exists by virtue of the normality of the space X). Define a decomposition σ_2^k of order k of the space X by putting $A_i^2 = A_i^1 \cup G$, $U_i^2 = U_i^1 \setminus G$, $V_i^2 = V_i^1 \setminus G$, $B^2 = B^1$, $C^2 = C^1 \cup G$, where $G = (X \setminus W(B^1))$. By Property 4.1, $\sigma_2^k \sim \sigma_1^k$. The decomposition σ^k, in which $A_i = A_i^2 \setminus \Phi$, $U_i = U_i^2 \cup \Phi$, $V_i = V_i^2$, $B = B^2$, $C = C^2 \setminus \Phi$, $\Phi = X \setminus [W(B^1)]$, is, by Property 4.1, equivalent to the decomposition σ_2^k; consequently, by transitivity, $\sigma^k \sim \sigma_1^k$ and, by construction, satisfies the required condition.

Property 4.3. Let σ^k be some decomposition of order k of the space X; let F be an arbitrary closed set in X, and $1 \le i \le k$; then there exists a decomposition $\sigma_{(i,F)}^k$ possessing the property that $\sigma_{(i,F)}^k \sim \sigma^k$, $\operatorname{Ind}(A^{(i,F)} \cap F) \le \operatorname{Ind} F - 1$, $\alpha_j^{(i,F)} = \alpha_j$, if $j \ne i$.

Indeed, it is easy to see that, without loss of generality, one may assume that $[U_i] \cap [V_i] = \emptyset$; then the closed sets $F \cap [U_i]$ and $F \cap [V_i]$ can be separated in F by a closed set D, with $\operatorname{Ind} D \le \operatorname{Ind} F - 1$. Construct on $F \cup [U_i] \cup [V_i]$ a real function $\varphi(x)$ possessing the property that $\varphi(x) = 0$ if and only if $x \in D$, $\varphi(x) = 1$ for $x \in [U_i]$, and $\varphi(x) = -1$ for $x \in [V_i]$; this is possible by Property 3.1. Let $f(x)$ be an extension of the function $\varphi(x)$ to the whole space X; then the elementary decomposition α_f, generated by the function $f(x)$, possesses the property that $A_f \subset A_i$, $U_f \supset U_i$, $V_f \supset V_i$, $A_f \cap F = D$. Putting $\alpha_j^{(i,F)} = \alpha_j$ if $j \ne i$, $\alpha_i^{(i,F)} = \alpha_f$, $B^{(i,F)} = B \cap A_f$, $C^{(i,F)} = C \cap A_f$, we obtain a partition $\sigma_{(i,F)}^k$, which, by property 4.1, is equivalent to σ^k and is the required one.

5. Group operation. The group operation is introduced in the set $p^k(X)$ on the basis of the following lemma:

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Lemma 1. Let $\operatorname{Ind} X = n$, and let s_1^k and s_2^k be two elements of the set $p^k(X)$, with $k \geq (n+1)/2$; then in the classes s_1^k and s_2^k there exist partitions $\bar{\sigma}_1^k$ and $\bar{\sigma}_2^k$ possessing the property that

$$\bigcup_{i=1}^{k}(\bar{A}_i^1 \cup \bar{V}_i^1) \cap \bigcup_{i=1}^{k}(\bar{A}_i^2 \cup \bar{V}_i^2) = \emptyset.$$

Proof. Let σ_1^k and σ_2^k be arbitrary partitions from the classes s_1^k and s_2^k, respectively. Successively applying property 4.3, it is easy to obtain partitions $\bar{\sigma}_1^k$ and $\bar{\sigma}_2^k$ possessing the property that

$$\operatorname{Ind} \bar{A}_1^1 \leq n-1, \quad \operatorname{Ind} \bar{A}_1^1 \cap \bar{A}_2^2 \leq n-2, \dots, \quad \operatorname{Ind} \bigcap_{i=1}^{k} \bar{A}_i^1 \leq n-k,$$

$$\operatorname{Ind}\Big(\bigcap_{i=1}^{k} \bar{A}_i^1\Big) \cap \bar{A}_1^2 \leq n-k-1, \dots, \quad \operatorname{Ind}\Big(\bigcap_{i=1}^{k} \bar{A}_i^1\Big) \cap \Big(\bigcap_{i=1}^{k} \bar{A}_i^2\Big) \leq n-2k \leq -1.$$

Consequently, $(\bar{B}^1 \cup \bar{C}^1) \cap (\bar{B}^2 \cup \bar{C}^2) = \emptyset$. Moreover, $\bar{B}^1 \cap \bar{B}^2 = \emptyset$, and, since the space X is normal, there exist disjoint neighborhoods $O(\bar{B}^1)$ and $O(\bar{B}^2)$ of the sets $\bar{B}^1$ and $\bar{B}^2$. From property 4.2 there follows the existence of partitions $\bar{\bar{\sigma}}_1^k \sim \bar{\sigma}_1^k$ and $\bar{\bar{\sigma}}_2^k \sim \bar{\sigma}_2^k$ possessing the property that

$$\bigcup_{i=1}^{k}(\bar{\bar{A}}_i^1 \cup \bar{\bar{V}}_i^1) \subset O(\bar{B}^1),$$

$$\bigcup_{i=1}^{k}(\bar{\bar{A}}_i^2 \cup \bar{\bar{V}}_i^2) \subset O(\bar{B}^2),$$

i.e. satisfying the conditions of the lemma.

Now let s_1^k and s_2^k be arbitrary elements of the set $p^k(X)$. Choose in the classes s_1^k and s_2^k elements σ_1^k and σ_2^k satisfying the condition of Lemma 1. The sequence $(\sigma_1^k + \sigma_2^k) = (\alpha_1, \dots, \alpha_k, B, C)$, where $A_i = A_i^1 \cup A_i^2$, $U_i = U_i^1 \cap U_i^2$, $V_i = V_i^1 \cup V_i^2$, $B = B^1 \cup B^2$, $C = C^1 \cup C^2$, is a partition of order k of the space X by virtue of the choice of the partitions σ_1^k and σ_2^k. The class containing the partition $(\sigma_1^k + \sigma_2^k)$ will be denoted by $(s_1^k + s_2^k)$. For $k > (n+1)/2$, it follows easily from Lemma 1 that $(s_1^k + s_2^k)$ does not depend on the accidental choice of σ_1^k and σ_2^k in the classes s_1^k and s_2^k.

6. The group $p^k(X)$. As is easy to see, the addition operation introduced in the set $p^k(X)$ is commutative. One can show that the zero class of the set $p^k(X)$ with respect to the introduced operation will be the class containing the partition $\sigma_0^k = (\alpha_1, \dots, \alpha_k, \emptyset, \emptyset)$, where $U_i = X$, $A_i = V_i = \emptyset$, $i = 1, \dots, k$, and that the class inverse to the class containing the partition $\sigma^k = (\alpha_1, \dots, \alpha_k, B, C)$ will be the class containing the partition $-\sigma^k = (\alpha_1, \dots, \alpha_k, C, B)$. The associativity of

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the introduced operation follows directly from the definition. Thus, the following holds:

Theorem 1. The set $p^k(X)$ is a group if $\operatorname{Ind} X < 2k-1$.

7. Connection with cohomotopy groups. We introduce the following notation for subsets of the $(k+1)$-dimensional coordinate space E^{k+1}: S_k is the unit k-dimensional sphere with center at the origin; S_i^{k-1} is the intersection of S^k with the plane $x_i = 0$; $E_i^{\delta k}$ is the intersection of S^k with the half-space $\delta x_i > 0$; $\delta = +1$ or -1; $a^\delta = (0, \dots, 0, \delta)$.

Let f be an arbitrary continuous mapping of the space X into S^k; then $\alpha_i^f = (f^{-1}(S_i^{k-1}), f^{-1}(E_i^{-k}), f^{-1}(E_i^{+k}))$ will be elementary partitions of the space X, and $\sigma_f^k = (\alpha_1^f, \dots, \alpha_k^f, f^{-1}(a^+), f^{-1}(a^-))$ will be a partition of order k of the space X.

Theorem 2. Let f and g be mappings of the space X into S^k that are homotopic to each other; then the partitions σ_f^k and σ_g^k are equivalent.

Proof. Since f and g are homotopic, there exists a mapping $F : (X \times I) \to S^k$ such that on the upper base it coincides with f, and on the lower—with g. Consider the decomposition of order k of the space $X \times I$ generated by the mapping F. It is easy to see that this decomposition realizes the equivalence of the decompositions σ_f^k and σ_g^k.

Theorem 3. Let X be a perfectly normal space and let σ^k be its decomposition of order k; then there exists a mapping $F : X \to S^k$ (unique up to homotopy) having the property that σ_F^k coincides with σ^k.

The proof of Theorem 3 is based entirely on Lemma 2.

Lemma 2. Let σ^k be a decomposition of order k of the space X, and let

$$f : \Xi \to \bigcup_{i=1}^{k} S_i^{k-1}$$

be a mapping having the property that

$$f^{-1}(S_i^{k-1} \cap E_j^{\delta k}) = \begin{cases} A_i \cap U_j, & \delta = +1, \\ A_i \cap V_j, & \delta = -1. \end{cases} \qquad (*)$$

Then there exists a mapping $F : X \to S^k$ such that σ_F coincides with σ^k.

The proof of Lemma 2 is carried out by a direct construction of the required mapping F.

Theorem 4. Let σ_1^k and σ_2^k be decompositions of order k of the space X, chosen according to Lemma 1 in the classes s_1^k and s_2^k. Let, further, f and g be mappings of the space X into S^k corresponding, according to Theorem 3, to

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the decompositions σ_1^k and σ_2^k. Then the mapping corresponding, according to Theorem 3, to the decomposition σ^k constructed in Lemma 1 is homotopic to the mapping $(f+g): X \to S^k$, where the sum $(f+g)$ is taken in the sense of Borsuk-Spanier ([1]).

Proof. Since the decompositions σ_1^k and σ_2^k satisfy the conditions of Lemma 1, there exists an elementary decomposition of the space

$$X: \alpha = (A, U, V),$$

satisfying the condition:

$$U \supset X \setminus \bigcap_{i=1}^{k} U_i^1, \qquad V \supset X \setminus \bigcap_{i=1}^{k} U_i^2.$$

Choose mappings f and g corresponding to the decompositions σ_1^k and σ_2^k and having the property that

$$f(A \cup U) = g(A \cup V) = \left(\frac{1}{\sqrt{k}}, \dots, \frac{1}{\sqrt{k}}, 0\right) = c.$$

Consider the mapping $F: X \to S_1^k \times S_2^k$, defined by the formula $F(x) = (f(x), g(x))$. It is easy to see that in the present case $F(x) \in S_1^k \times c$ if $x \in A \cup U$, and $F(x) \in S_2^k \times c$ if $x \in A \cup V$. Consequently, the mapping F is a mapping into the bouquet of spheres $S_1^k \vee S_2^k$. If we identify the spheres S_1^k and S_2^k, then we obtain the mapping $(f+g): X \to S^k$ in the sense of Borsuk-Spanier. From the construction it is clear that $\sigma_{(f+g)}^k = \sigma^k$, and Theorem 4 is proved.

Theorem 5. The group $P^k(X)$ is isomorphic to the k-th cohomotopy group $\pi^k(X)$ of the space X.

The proof follows directly from Theorems 2, 3, and 4.

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1. E. Spanier, Ann. of Math., **50**, 203 (1949).

Note: Figure translations are in progress. See original paper for figures.

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