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Abstract

Full Text

Hydromechanics

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On Self-Similar Solutions of the Equations of the Laminar Boundary Layer in Magnetohydrodynamics

1°. Recently numerous works have appeared devoted to finding self-similar solutions of the equations of laminar flow of a viscous electrically conducting liquid or gas in the presence of a magnetic field. Most of them investigate flow past a plate or flow in the vicinity of the critical point of a blunt body under various restrictions: for example, the interaction of the magnetic field with the inviscid flow is not taken into account; in the energy equation terms characterizing viscous and Joule dissipation are omitted; the variation of the intensity B of the magnetic field along and across the flow is not taken into account. Other self-similar solutions were obtained in work ([1]) for the case when the velocity outside the boundary layer and the magnetic-field intensity vary according to power laws related to one another along the body in suitably chosen variables (and not in the original "physical" plane). In work ([2]) it was shown that the quantity B in the vicinity of the critical point decreases, in norm, inversely proportionally to the cube of the distance, and the influence of the magnetic field on the inviscid flow was established. This gave occasion to investigate in work ([3]) the influence of the indicated variation of the quantity B on heat transfer. In doing so, the self-similar equations of work ([1]) were used, in which the quantity B was likewise taken to be constant, equal to its mean value between the critical point and the shock wave.

It is not difficult, however, to obtain self-similar solutions also when B varies according to a power law both along the body and along the normal. Here, in addition, several other self-similar solutions of the equations of the laminar boundary layer in magnetohydrodynamics are indicated and, in a certain sense, all possible cases of obtaining such solutions are exhausted. The present work is devoted to these questions. The cases of an incompressible liquid and a gas are investigated.

2°. In the case of an incompressible liquid the equations of the laminar boundary layer may be written in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{\rho}B^2 u; \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0;$$

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$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\frac{\nu}{\mathrm{Pr}}\frac{\partial^2\theta}{\partial y^2}+\nu\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma}{\rho}B^2u^2. \tag{1}$$

Here $\theta(x,y)$ is the difference between the enthalpy of unit mass at any point and in the external stream, while the last two terms in the energy equation characterize, respectively, viscous and Joule dissipation. The boundary conditions are

$$u(x,0)=v(x,0)=0;\qquad u(x,\infty)=U_e(x);\qquad \theta(x,0)=\tau(x);\qquad \theta(x,\infty)=0. \tag{2}$$

Let us first of all define the function $U(x)$ by the equality

$$-\frac{1}{\rho}\frac{dp}{dx}=UU'.$$

In the presence of body forces created by the magnetic field, this function

in the external flow, different from the velocity distribution $U_e(x)$ at the boundary of the boundary layer. We shall now seek those distributions $U(x)$ and $\tau(x)$ for which the equations of system (1) are reduced to ordinary ones by means of similarity transformations over the velocity and enthalpy profiles (for a detailed definition of self-similarity, see (4-6)), and in doing so find the relation between $U(x)$ and $U_e(x)$. In accordance with the results of (4, 5), for this it is necessary to set

$$u(x,y)=U\varphi'(\xi);\qquad \xi=yK(x);\qquad \theta(x,y)=\tau(x)g(\xi);$$

$$v(x,y)=\left\{\frac{UK'}{K^2}-\frac{U'}{K}\right\}\varphi(\xi)-\frac{UK'}{K}\xi\varphi'(\xi). \tag{3}$$

Here the continuity equation is satisfied identically, while the two other equations of system (1) take the form

$$\varphi'^2-\left\{1-\frac{UK'}{U'K}\right\}\varphi\varphi''=1+\nu\frac{K^2}{U'}\varphi'''-\frac{\sigma}{\rho}\frac{B^2}{U'}\varphi';$$

$$\frac{U\tau'}{U'\tau}g\varphi'-\left\{1-\frac{UK'}{U'K}\right\}g'\varphi=\frac{\nu}{\mathrm{Pr}}\frac{K^2}{U'}g''+\frac{\sigma}{\rho}\frac{B^2}{U'}\frac{U^2}{\tau}\varphi'^2+\nu\frac{U^2}{\tau}\frac{K^2}{U'}\varphi''^2. \tag{4}$$

It is easy to see that these equations will be ordinary if and only if (with the corresponding normalization of the function $K(x)$)

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$$1 - UK'/U'K = 1/\beta; \qquad \nu K^2/U' = 1/\beta; \qquad U^2/\tau = \alpha; \qquad U\tau'/U'\tau = \gamma, \tag{5}$$

where α, β, γ are constants and $\gamma = 2$ for $\alpha \neq 0$, while the quantity $\dfrac{\sigma}{\rho}\dfrac{B^2}{U'}$ is a function of ξ. As in (4, 5), one can verify that (5) is fulfilled when

$$U \sim x^n \quad (\beta = 2n/(n+1)) \qquad \text{or} \qquad U \sim e^{ax} \quad \text{for } \beta = 2, \tag{6}$$

and $\tau = \dfrac{1}{\alpha}U^2$. Regarding the quantity B, let us assume that at the wall it is a function $B_w(x)$, and in the y-direction it varies inversely proportionally to the s-th power of y. In addition, it should be borne in mind that at the outer boundary of the boundary layer (i.e., as $y \to \infty$) the field intensity B may differ from zero, being a function $B_e(x)$. Therefore set

$$B(x, y) - B_e(x) = b(x)[a(x) + y]^{-s}.$$

Since $y = \xi K^{-1}(x)$, it is easy to choose $a(x)$ and $b(x)$ so that the wall conditions are satisfied:

$$B(x, y) = B_e(x) + \frac{[B_w(x) - B_e(x)]}{(1+\xi)^s}.$$

We shall also assume that $B_e(x)/B_w(x) = m$—a quantity characterizing the degree of attenuation of the field intensity across the flow—does not depend on x. To obtain self-similar solutions it is now necessary that the magnetic parameter

$$\zeta = \frac{\sigma}{\rho}\frac{B_w^2}{U'}\beta$$

be constant. This is possible when, in accordance with (6), $B_w \sim \sqrt{U'}$, i.e. $B_w \sim x^{(n-1)/2}$ or $B_w \sim e^{ax/2}$.

In this case equations (4) and boundary conditions (2) take the form

$$\varphi''' + \varphi\varphi'' = \beta(\varphi'^2 - 1) + \zeta\varphi' \left\{ m + \frac{1-m}{(1+\xi)^s} \right\}^2; \tag{7}$$

$$\frac{1}{\mathrm{Pr}} g'' + g'\varphi - \beta\gamma g\varphi' + \alpha\varphi''^2 + \alpha\zeta\varphi'^2 \left\{ m + \frac{1-m}{(1+\xi)^s} \right\}^2 = 0; \tag{8}$$

$$\varphi(0) = \varphi'(0) = 0; \qquad \varphi'(\infty) = C; \qquad g(0) = 1; \qquad g(\infty) = 0. \tag{9}$$

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The constant $C > 0$ is determined from (7) as $\xi \to \infty$, i.e., from the equation $\beta(C^2 - 1) + \zeta m^2 C = 0$. Moreover, $U_e(x) = CU(x)$. Note that in the absence of interaction of the field with the external flow, $B_e = m = 0$.

Then $C = 1$ and $U_e = U$. If viscous and Joule dissipation are neglected, then in (8) one must set $\alpha = 0$. In this case new self-similar solutions are obtained, when $\tau \sim U^\gamma$ for any γ (and not only $\gamma = 2$, as was the case when the dissipative terms were taken into account).

Remark. In deriving equations (4), division by U' was performed. In doing so, of course, the case of a plate $U = \text{const}$ was omitted. The subsequent analysis shows that in this case the self-similar equations are obtained from (7) and (8), if one sets $\beta = 0$. In this case necessarily

$$\tau = \text{const} \quad \text{and} \quad \xi = \frac{2\sigma}{\rho U} \frac{B_w^2}{x^{-1}}, \quad \text{i.e.} \quad B_w \sim x^{-1/2}.$$

If, in the case of a plate, the dissipative terms are absent, then the energy equation takes the form $(1/\mathrm{Pr})g'' + g'\varphi - \gamma g\varphi' = 0$. In this case $\tau(x)$ may be an arbitrary power function $\tau \sim x^{\gamma/2}$.

Let us also note that here the case $n = -1$, $\beta = \infty$, i.e. $1 - UK'/U'K = 0$, which corresponds to flow in a diffuser, has remained unconsidered. This case is easily analyzed by putting $\nu K^2/U' = 1$ in equalities (5). Here too the conditions of self-similarity are easily obtained.

3°. The investigation carried out pertains to the case of an incompressible fluid with constant conductivity σ. These assumptions are quite admissible in a number of cases, in particular in studying the flow near the critical point of a blunt body moving with a large supersonic velocity. It is precisely in this case that in paper ([2]) the interaction of a magnetic field with an inviscid flow was investigated.

We shall now indicate a way of obtaining analogous results in the case of a compressible fluid. We shall use the results of paper ([6]). We write the initial boundary-layer equations in the form

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \sigma B^2 u;$$

$$\rho\left(u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y}\right) = \frac{1}{\mathrm{Pr}}\frac{\partial}{\partial y}\left(\mu\frac{\partial \theta}{\partial y}\right); \qquad \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0. \tag{10}$$

Here, for simplicity, the energy equation is for the time being written in the simplest form, without allowance for Joule dissipation and for Prandtl numbers Pr close to unity. Moreover, as in ([6]), $\theta = T + u^2/2Ec_p$, $\mu/\mu_0 = T/T_0$ (T_0 is the stagnation temperature). Similarly to how this was done in (7), we introduce new variables analogous to Dorodnitsyn' s variables:

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$$X = \int_0^x \left(\frac{p}{p_0}\right)^r dx; \qquad Y = \int_0^y \frac{\rho}{\rho_0}\, dy. \tag{11}$$

Here r is as yet arbitrary. We now define the new function $U(x)$ by the equality

$$\frac{p}{p_0} = \left[1 - \left(\frac{U}{U_*}\right)^2\right]^{k/(k-1)}, \qquad U_* = \frac{2k}{k+1}\, gRT_0.$$

This function, generally speaking, differs from the velocity U_e outside the layer.

Next we denote

$$V = u(p_0/p)^r \frac{\partial X}{\partial x} + v(p_0/p)^{r-1}\frac{T_0}{T},$$

and introduce dimensionless variables, referring, as in paper ([6]), the velocities u and U to U_*, and defining the dimensionless temperature as $\theta/T_0 - 1$. We also introduce the dimensionless conductivity and magnetic-field intensity.

Moreover, as in ([1]), we shall assume that at large supersonic speeds of motion of the body (and subsonic speeds in the boundary layer) $\rho_e/\rho = \theta/T_0$. After this, system (10) is reduced to the following:

$$u\frac{\partial u}{\partial X} + V\frac{\partial u}{\partial Y} = (\theta+1-u^2)\frac{UU'}{1-U^2} + (1-U^2)^{-\delta}\frac{\partial^2 u}{\partial Y^2} - \frac{\sigma B^2}{\rho_0}\, u(\theta+1)(1-U^2)^{-\delta-1}; \tag{12}$$

$$\frac{\partial u}{\partial X} + \frac{\partial V}{\partial Y} = 0; \qquad u = \frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} = (1-U^2)^{-\delta}\frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial Y^2}; \qquad \delta = \frac{k(\alpha-1)}{k-1};$$

where all quantities are dimensionless, and θ is now already the dimensionless temperature.* The boundary conditions in the variables X, Y can be written as equalities (2), and the self-similar solutions can be sought by means of relations (3).

It can be shown that self-similar solutions can be obtained either for a constant wall temperature ($\tau = \text{const}$), or for a constant velocity U (the plate case). In the first case ($\tau = \text{const}$), for self-similarity it is necessary that, as in ([6]),

$$(1-U^2)\left(1 - \frac{UK'}{U'K}\right) = -\frac{K^2}{U'}(1-U^2)^{-\delta+1} = \frac{1}{\beta}. \tag{13}$$

Choosing now r so that $\delta - 1 = 1/\beta$, and eliminating K from the equalities (13), we find that the solutions of the resulting differential equation for $U(X)$ are the

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functions specified by equalities (6), where X should be written instead of x. The desired system of equations then takes the form

$$\varphi''' + \varphi\varphi'' = \beta(\varphi'^2 - 1 - \tau g) + \zeta\varphi'(1+\tau g)\left\{m + \frac{1-m}{(1+\xi)^s}\right\}^2; \quad g'' + \Pr\varphi g' = 0. \tag{14}$$

In this case, however, the expression for the magnetic parameter ζ is more complicated:

$$\zeta = \frac{\sigma B_w^2}{\rho_0}\frac{\beta}{U'}(1-U^2)^{-1-1/\beta}; \qquad B_w \sim \{U'(1-U^2)^{1+1/\beta}\}^{1/2}. \tag{15}$$

If, in accordance with the assumption of work (1) mentioned above, it is assumed that the dimensionless velocity U satisfies the condition $U^2 \ll 1$ (in dimensional form $(U/U_*)^2 \ll 1$), then from (15) we obtain $B_w \sim \sqrt{U'}$. In other words, the law relating B_w to U' is the same for a gas and for an incompressible fluid. The boundary conditions of system (14) are conditions (9). The value of the constant C and the relation between U_e and U will be the same and are established in the same way as for an incompressible fluid.

Analogously to the preceding, one can also consider the second possible case of obtaining self-similar solutions—the plate case ($U = \text{const}$). The desired solutions, as for an incompressible fluid, are obtained for an arbitrary power-law distribution of the wall temperature $\tau(X)$. In addition, by putting simultaneously $U = \text{const}$ and $\tau = \text{const}$, one can obtain self-similar solutions also when Joule dissipation and the additional term

$$\left(1 - \frac{1}{\Pr}\right)\frac{\partial}{\partial y}\left(\frac{\mu}{Ec_p}\, u\,\frac{\partial u}{\partial y}\right)$$

are taken into account in the energy equation.

Remark. Of course, self-similar solutions can also be obtained for any other choice of the parameter r in the change of variables (11). However, for another choice of r, the solution of system (13) will no longer be the functions (6), but some other functions, which are not difficult to find. The form of the functions $U(X)$ that give self-similar solutions depends on the change of variables (11), i.e., on the choice of the parameter r. If we return to the original variables x, y, we obtain the same $U(x)$ for any r. The form of equations (14) also does not depend on r.

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* Let us note that in work ([6]), in the equation analogous to the first equation of system (12), instead of the factor $\theta+1-u^2$ it is erroneously written $\theta-u^2$. The corresponding correction should also be made in the final form of the equations of work ([6]).

Note: Figure translations are in progress. See original paper for figures.

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