MATHEMATICAL PHYSICS

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Abstract

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CANONICAL TRANSFORMATIONS OF OPERATORS IN THE SECOND-QUANTIZATION REPRESENTATION

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1. Let $\hat{c}(\xi)$, $\hat{c}^*(\xi)$ be operators acting in a Hilbert space H. Let the parameter ξ, which enumerates the operators, range over a certain set Ξ endowed with a measure $d\xi$. (In cases of practical significance, Ξ is the product of a finite-dimensional linear space X, endowed with Lebesgue measure, by a finite or countable set.) Suppose that the commutation relations
$$[\hat{c}(\xi),\, \hat{c}^*(\xi')] = \delta(\xi,\xi') \tag{1}$$
or
$$\{\hat{c}(\xi),\, \hat{c}^*(\xi')\} = \delta(\xi,\xi') \tag{2}$$
are satisfied between the operators $\hat{c}$, $\hat{c}^*$.

Here, as usual, $[a,b] = ab - ba$, $\{a,b\} = ab + ba$; $\delta(\xi,\xi')$ is the Dirac δ-function:
$$\int \delta(\xi,\xi')f(\xi')\,d\xi' = f(\xi).$$

The case in which the operators $\hat{c}$, $\hat{c}^*$ satisfy relations (1) will be called **bosonic**; the case in which the operators $\hat{c}$, $\hat{c}^*$ satisfy relations (2) will be called **fermionic**.*

2. Let $\hat{A}$ be a linear operator in H, representable in the form
$$\hat{A} = \sum \int K(\xi_1,\dots,\xi_n \mid \eta_1,\dots,\eta_p)\, \hat{c}^*(\xi_1)\cdots\hat{c}^*(\xi_n)\hat{c}(\eta_1)\cdots\hat{c}(\eta_p)\,d\xi_1\cdots d\eta_p. \tag{3}$$
Consider the linear canonical transformation
$$\begin{aligned}\hat{c}(\xi) &= \int \Phi(\xi,\eta)\hat{a}(\eta)\,d\eta + \int \Psi(\xi,\eta)\hat{a}^*(\eta)\,d\eta,\\ \hat{c}^*(\xi) &= \int \overline{\Psi}(\xi,\eta)\hat{a}(\eta)\,d\eta + \int \overline{\Phi}(\xi,\eta)\hat{a}^*(\eta)\,d\eta.\end{aligned} \tag{4}$$

Substituting (4) into (3) and using the permutation relations between a, a^*, one can write $\hat{A}$ in the form

$$\hat{A} = \sum \int \widetilde{K}(\xi_1, \dots, \xi_n \mid \eta_1, \dots, \eta_p)\, \hat{a}^*(\xi_1) \cdots \hat{a}^*(\xi_n) \hat{a}(\eta_1) \cdots \hat{a}(\eta_p)\, d\xi_1 \cdots d\eta_p. \tag{5}$$

Obtaining formula (5) from (4) and (3) in the manner described is associated with cumbersome computations that are difficult to survey. The purpose of the present work is to establish the connection between (3) and (5), bypassing these computations.

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* How exactly the space H and the operators $\hat{c}$, $\hat{c}^*$ should be constructed so that relations (1) or (2) hold is not important for us.

3. The Bose case

To each operator $\hat{A}$ written in the form (3), let us assign the functional

$$A(c^*, c) = \sum \int K(\xi_1, \dots, \xi_n \mid \eta_1, \dots, \eta_p) c^*(\xi_1) \dots c^*(\xi_n) c(\eta_1) \dots c(\eta_p)\, d\xi_1 \dots d\eta_p; \tag{6}$$

$c^*(\xi)$, $c(\xi)$ are some complex-valued functions on the set Ξ. To the same operator, written in the form (5), we assign the analogous functional

$$\widetilde{A}(a^*, a) = \sum \int \widetilde{K}(\xi_1, \dots, \xi_n \mid \eta_1, \dots, \eta_p) a^*(\xi_1) \dots a^*(\xi_n) a(\eta_1) \dots$$

$$\dots a(\eta_p)\, d\xi_1 \dots d\eta_p.$$

It is clear that the correspondence between the functionals A and $\widetilde{A}$ is linear. By analogy with the finite-dimensional case, we shall seek it in the form

$$\widetilde{A}(a^*, a) = \int \mathcal{K}(a^*, a \mid c^*, c)\, A(c^*, c) \prod_{\xi} dc^*(\xi)\, dc(\xi); \tag{7}$$

$$\prod_{\xi} dc^*(\xi)\, dc(\xi)$$

is the symbol of continual integration. Our aim is to compute $\mathcal{K}$.

Theorem 1. *The kernel $\mathcal{K}$ is given by the formula*

$$\mathcal{K} = (\det \pi \Psi^* \Psi)^{-1/2} \exp\left\{-\frac{1}{2}\left[(\bar{\Phi}\Psi^{-1} b, b) + (\Phi \Psi^{-1} b^*, b^*) - 2(b^*, b)\right]\right\}, \tag{8}$$

where $\Psi, \bar{\Psi}, \Phi, \bar{\Phi}$ *are operators defined by the kernels* $\Psi(\xi, \eta)$, $\bar{\Psi}(\xi, \eta)$, $\Phi(\xi, \eta)$, $\bar{\Phi}(\xi, \eta)$, *respectively;*

$$b(\xi) = \int \Phi(\xi, \eta) a(\eta)\, d\eta + \int \Psi(\xi, \eta) a^*(\eta)\, d\eta - c(\xi);$$

$$b^*(\xi) = \int \bar{\Psi}(\xi, \eta) a(\eta)\, d\eta + \int \bar{\Phi}(\xi, \eta) a^*(r)\, dr - c^*(\xi); \qquad (b_1, b_2) = \int b_1(\xi) b_2(\xi)\, d\xi.$$

The integration in (7) is performed over a contour along which

$$\operatorname{Re}\left[(\bar{\Phi}\Psi^{-1}c, c) + (\Phi\bar{\Psi}^{-1}c^*, c^*)\right] > 0 \;\; * .$$

Let us give a sketch of the computation of the kernel $\mathscr{K}$ for the case when Ξ is a finite set consisting of n elements. In this case $A(c^*, c)$, $\widetilde{A}(a^*, a)$ are functions of $2n$ variables. Introduce in the space of functions $A(c^*, c)$ the operators $c^*_\ell(\xi)$, $c_\ell(\xi)$, $c^*_{\text{pr}}(\xi)$, $c_{\text{pr}}(\xi)$, arranged so that the function $c^*_\ell(\xi)A$ corresponds to the operator $\hat{c}^*(\xi)\hat{A}$, the function $c_\ell(\xi)A$ to the operator $\hat{c}_\ell(\xi)\hat{A}$, and so on. A simple calculation shows that the operators c^*_ℓ, c_ℓ, c^*_{pr}, c_{pr} are given by the formulas

$$c^*_{\text{pr}}(\xi)A = \left(\frac{\partial}{\partial c(\xi)} + c^*(\xi)\right)A; \qquad c_{\text{pr}}(\xi)A = c(\xi)A;$$

$$c^*_\ell(\xi)A = c^*(\xi)A; \qquad c_\ell(\xi)A = \left(\frac{\partial}{\partial c^*(\xi)} + c(\xi)\right)A. \tag{9}$$

We shall consider analogous operators in the space of functions $\widetilde{A}(a^*, a)$.

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* This condition does not determine the path of integration uniquely. We shall describe in more detail an important class of integration paths. For this purpose represent $A = \bar{\Phi}\Psi^{-1}$ in the form $A = A_1 + iA_2$, and $c(\xi), c^*(\xi)$ in the form $c(\xi) = x(\xi) + iy(\xi)$, $c^*(\xi) = \tilde{x}(\xi) + i\tilde{y}(\xi)$; $x, y, \tilde{x}, \tilde{y}$ are real functions; A_1, A_2 are Hermitian operators mapping real functions to real ones. Parametrize the path of integration by putting $x = \alpha t$, $y = \beta t$, $\tilde{x} = \alpha\tau$, $\tilde{y} = \beta\tau$. Here $t = t(\xi)$, $\tau = \tau(\xi)$ are real functions; α, β are certain operators. Then

$$\prod dc\, dc^* = |\det(\alpha + i\beta)|^2 \prod dt\, d\tau,$$

$$\operatorname{Re}\left[(\bar{\Phi}\Psi^{-1}c, c) + (\Phi\bar{\Psi}^{-1}c^*, c^*)\right] = ((\alpha' A_1 \alpha - \beta' A_1 \beta - \alpha' A_2 \beta - \beta' A_2 \alpha)t, t) + ((\alpha' A_1 \alpha - \beta' A_1 \beta - \alpha' A_2 \beta - \beta' A_2$$

Thus one may integrate over any contour of the described form, determined by operators α, β satisfying the conditions:

$$1) \quad \alpha' A_1 \alpha - \beta' A_1 \beta - \alpha' A_2 \beta - \beta' A_2 \alpha > 0; \qquad 2) \quad 0 < |\det(\alpha + i\beta)|^2 < \infty.$$

Substitute $c_\lambda A$ in (7) in place of A. Taking (4) into account, we have*

$$\sum (\Phi(\xi,\eta) a_\lambda(\eta) + \Psi(\xi,\eta) a_\lambda^*(\eta)) \widetilde{A} = \int \mathcal{K}(a^*, a \mid c^*, c) c_\lambda(\xi) A \prod dc^* dc. \quad (10)$$

On the other hand,

$$\sum (\Phi(\xi,\eta) a_\lambda(\eta) + \Psi(\xi,\eta) a_\lambda^*(\eta)) \widetilde{A} = \int \sum (\Phi(\xi,\eta) a_\lambda(\eta) + \Psi(\xi,\eta) a_\lambda^*(\eta)) \times \mathcal{K} A \prod dc^* dc. \quad (10')$$

Substitute in (10) $c_\lambda(\xi)$ from (9) and integrate by parts. Comparing the result obtained with $(10')$ and replacing a_λ, a_λ^* by their explicit expressions, we obtain for $\mathcal{K}$ the equation

$$\sum \left[\Phi(\xi,\eta) \left(\frac{\partial}{\partial a^*(\eta)} + a(\eta) \right) + \Psi(\xi,\eta) a^*(\eta) \right] \mathcal{K} = \left(c(\xi) - \frac{\partial}{\partial c^*(\xi)} \right) \mathcal{K}. \quad (11a)$$

In an analogous way, substituting in (7), in place of A, $c_\lambda^* A$, $c_{\text{pr}} A$, $c_{\text{pr}}^* A$, we obtain three more equations for $\mathcal{K}$:

$$\sum \left[\overline{\Psi}(\xi,\eta) \frac{\partial}{\partial a^*(\eta)} + \overline{\Psi}(\xi,\eta) a(\eta) + \overline{\Phi}(\xi,\eta) a^*(\eta) \right] \mathcal{K} = c^*(\xi) \mathcal{K}; \quad (11b)$$

$$\sum \left[\Psi(\xi,\eta) \frac{\partial}{\partial a(\eta)} + \Phi(\xi,\eta) a(\eta) + \Psi(\xi,\eta) a^*(\eta) \right] \mathcal{K} = c(\xi) \mathcal{K}; \quad (11c)$$

$$\sum \left[\overline{\Phi}(\xi,\eta) \frac{\partial}{\partial a(\eta)} + \overline{\Psi}(\xi,\eta) a(\eta) + \overline{\Phi}(\xi,\eta) a^*(\eta) \right] \mathcal{K} = \left(-\frac{\partial}{\partial c(\xi)} + c^*(\xi) \right) \mathcal{K}. \quad (11d)$$

In addition to equations (11), $\mathcal{K}$ satisfies the normalization condition

$$\int \mathcal{K}(a^*, a \mid c^*, c) \prod dc^* dc = 1. \quad (11')$$

Condition (11′) is the expression of the fact that, under a canonical transformation, the identity operator passes into the identity. Equations (11), together with condition (11′), have a unique solution.

The path of integration in (7) is determined by two conditions: first, the integral (7) must exist for a sufficiently large class of functions A representing operators; second, the integration-by-parts formula must be valid,

$$\int \mathcal{K} \frac{\partial A}{\partial c(\xi)} \prod dc^* dc = -\int \frac{\partial \mathcal{K}}{\partial c(\xi)} A \prod dc^* dc;$$

$$\int \mathcal{K} \frac{\partial A}{\partial c^*(\xi)} \prod dc^* dc = -\int \frac{\partial \mathcal{K}}{\partial c^*(\xi)} A \prod dc^* dc.$$

4. The fermion case. Consider the Grassmann algebra G, generated by the elements $c(\xi), c^*(\xi)$, i.e., the algebra with generators $c(\xi), c^*(\xi)$ satisfying the relations

$$\{c(\xi), c(\xi')\} = \{c(\xi), c^*(\xi')\} = \{c^*(\xi), c^*(\xi')\} = 0.$$

To each operator $\widehat{A}$ given by formula (3), we put in correspondence the element G

$$A(c^*, c) = \sum \int K(\xi_1, \dots, \xi_n \mid \eta_1, \dots, \eta_p) c^*(\xi_1) \dots c^*(\xi_n) c(\eta_1) \dots c(\eta_p) \, d\xi_1 \dots d\eta_p.$$

Reasoning analogous to that carried out in the preceding subsection shows that a canonical transformation generates a linear transformation in G. We shall seek this transformation in the form

$$\widetilde{A}(a^*, a) = \int^F \mathcal{K}(a^*, a \mid c^*, c) A(c^*, c) \prod_\xi dc^*(\xi) \, dc(\xi). \tag{12}$$

* Since the set Ξ is assumed finite, the integral in (4) is replaced here by a sum.

Before describing the kernel $\mathcal{K}$, let us define $\int \cdots \prod dc^* dc$. Let first $\widetilde{G}$ be a finite Grassmann algebra with generators $x_1, \dots, x_n$. Introduce the symbols $dx_1 \dots dx_n$ and subject them to the commutation relations: $\{dx_i, dx_k\} = \{dx_i, x_k\} = 0$. Next put, by definition,

$$\int^F dx_k = 0, \qquad \int^F x_k \, dx_k = 1. \tag{13}$$

A multiple integral will be understood as an iterated one. The integral on the algebra $\widetilde{G}$ thus defined will be called a **Fermi integral***.

The continual integral in (12) will be understood as the limit of n-fold Fermi integrals, just as an ordinary continual integral is understood as the limit of ordinary n-fold integrals.

Theorem 2. *In the Fermi case the kernel $\mathcal{K}$ is equal to*

$$\mathcal{K} = (\det \Psi\Psi^{[*]})^{1/2} \exp \frac{1}{2}\left[(\Phi\overline{\Psi}^{-1}b, b) - (\Phi\overline{\Psi}^{-1}b^*, b^*) + 2(b^*, b)\right] \tag{14}$$

(the notation is the same as in (7)).

We give a sketch of the calculation for the case when Ξ is a finite set and, consequently, G is a finite Grassmann algebra. Let first $\widetilde{G}$ be an arbitrary finite Grassmann algebra with generators $x_1, \dots, x_n$. For arbitrary $F \in \widetilde{G}$ define the left and right derivatives $\dfrac{\partial}{\partial x_k}F$ and $F\dfrac{\partial}{\partial x_k}$ as follows: if $F = x_{i_1} \dots x_{i_l}$, $x_k \neq x_{i_s}$ for any s, then $\dfrac{\partial}{\partial x_k}F = F\dfrac{\partial}{\partial x_k} = 0$. If $F = x_{i_1} \dots x_{i_{s-1}} x_k x_{i_{s+1}} \dots x_{i_l}$, then

$$\frac{\partial}{\partial x_k}F = (-1)^{s+1}x_{i_1} \dots x_{i_{s-1}}x_{i_{s+1}} \dots x_{i_l}, \qquad F\frac{\partial}{\partial x_k} = (-1)^{l-s}x_{i_1} \dots x_{i_{s-1}}x_{i_{s+1}} \dots x_{i_l}.$$

The left and right derivatives are extended to linear combinations of monomials in the usual way.

The integration-by-parts formulas hold:

$$\int^F F\left(\frac{\partial}{\partial x_k}G\right)dx_1 \dots dx_n = \int^F \left(F\frac{\partial}{\partial x_k}\right)G\,dx_1 \dots dx_n,$$

$$\int^F F\left(G\frac{\partial}{\partial x_k}\right)dx_1 \dots dx_n = -\int^F \left(\frac{\partial}{\partial x_k}F\right)G\,dx_1 \dots dx_n. \tag{15}$$

We return to our problem. First of all, consider the operators $c_\ell^*, c_{\mathrm{pr}}^*, c_\ell, c_{\mathrm{pr}}$, analogous to the corresponding operators in the Bose case. These operators turn out to be equal to

$$c_\ell^*(\xi)F = c^*(\xi)F, \qquad c_{\mathrm{pr}}^*(\xi)F = F\left(\frac{\partial}{\partial c(\xi)} + c^*(\xi)\right),$$

$$c_\ell(\xi)F = \left(\frac{\partial}{\partial c^*(\xi)} + c(\xi)\right)F, \qquad c_{\mathrm{pr}}(\xi)F = Fc(\xi). \tag{16}$$

Using (16) and the integration-by-parts formulas (15), we compose equations for the kernel, proceeding from the same considerations as in the Bose case. The equations obtained are almost indistinguishable from (11). In addition to these equations, condition (11′) remains in force. The equations together with this condition determine $\mathcal{K}$ uniquely.

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References

1. I. M. **Khalatnikov**, ZhETF, **28**, 635 (1955).

[*] For an equivalent, but different in form, definition of the Fermi integral, see (1).

Note: Figure translations are in progress. See original paper for figures.

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