M. I. VISHIK and Corresponding Member of the Academy of Sciences of the USSR L. A. LYUSTERNIK

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Abstract

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MATHEMATICS

M. I. VISHIK and Corresponding Member of the Academy of Sciences of the USSR L. A. LYUSTERNIK

ON THE INITIAL JUMP FOR NONLINEAR DIFFERENTIAL EQUATIONS CONTAINING A SMALL PARAMETER

For simplicity, we consider the Cauchy problem for a nonlinear differential equation of second order

$$L_\varepsilon y \equiv \varepsilon y'' + \varphi(x, y, y') = 0, \qquad y\big|_{x=0} = y_0, \qquad y'\big|_{x=0} = C/\varepsilon^\beta \quad (\beta > 0) \quad (1)$$

and investigate, as $\varepsilon \to 0$, the geometric limit of the integral curves y_ε of this problem, which may contain a segment of the axis Oy: $[y_0, y_0 + B]$. We say that in this case the phenomenon of an initial jump takes place. For a quasilinear equation this phenomenon was considered in ([1]). We shall assume that $\varphi(x, y, y')$ grows as $y' \to \infty$ like $|y'|^l$, $0 < l \leq 2$ (it is precisely under these conditions that an initial jump is possible).

Let us first consider the case when $l = 1 + \alpha$, $0 < \alpha < 1$. To obtain an asymptotic expansion of arbitrary order for the solution $y_\varepsilon = y_\varepsilon(x)$ of problem (1), we shall assume that for large y', $y' > N$ (and an analogous expansion for $y' < -N$),

$$\varphi(x, y, y') = (y')^{1+\alpha} \left[\psi_{00}(x, y) + \sum_{i,j}{}' \psi_{ij}(x, y)(y')^{-ik-j} \right], \qquad k > 0, \quad (2)$$

$$\psi_{00}(y) \geq a^2 > 0;$$

the sum $\sum'$ is taken over indices $i \geq 0$, $j \geq 0$, $i + j > 0$; this sum may be finite or infinite in one or both indices (in this case convergence is understood in the asymptotic sense); finally, by such a sum we also mean a finite sum with a remainder term. In particular, for example, we may have $\varphi(x, y, y') = (y')^{1+\alpha}[\psi_{00}(x, y) + O(1)y'^{-r}]$, $r > 0$, or, for example, when $y' > 0$,

$$\varphi(x, y, y') = y'^{\,1+\alpha} \frac{y'}{\sqrt{1 + (y')^2}} + ay' = y'^{\,1+\alpha} \left[1 + \sum C_i (y')^{-2i} + ay'^{-\alpha}\right].$$

We note that more complicated expansions of the function φ of type (2) will affect only the estimates. Investigation of the first approximation, when φ is replaced by $\psi_{00}(0,y){y'}^{1+\alpha}$, shows that the jump phenomenon will take place if in (1) one sets

$$y'\big|_{x=0} = \frac{C}{\varepsilon^{\beta}},$$

where $\beta = (1-\alpha)^{-1}$ (for definiteness we take $C > 0$). In this case the length B of the initial jump is determined from the equation

$$C^{1-\alpha} = (1-\alpha)\int_{y_0}^{y_0+B} \psi_{00}(0,y)\,dy. \tag{3}$$

To construct the asymptotic expansion of y_ε, it is expedient (cf., for example, the works ([2,3])) to divide the region of rapid variation of y into two zones. The third zone is the remaining part of the half-axis $x > 0$.

In the first zone—the zone of the principal part of the jump, y_ε—the variable y changes by a finite amount close to B, while x changes by an amount of order $O(\varepsilon^{\gamma})$, $\gamma > 1$. Therefore it is expedient in (1) to take y as the argument, and $x = x(y)$; then equation (1), by virtue of (2), takes the form

$$\varepsilon\frac{d^2x}{dy^2} = \left(\frac{dx}{dy}\right)^{2-\alpha}\left[\psi_{00}(x,y) + \sum_{r,p}\psi_{pr}(x,y)\left(\frac{dx}{dy}\right)^{pk+r}\right]. \tag{4}$$

Representing $\psi_{pr}(x,y)$ in the form

$$\psi_{pr}(x,y) = \psi_{pr0}(y) + \sum_{s\geqslant 1} x^s\psi_{prs}(y) \tag{5}$$

and making the change of variables

$$x = \varepsilon^{\beta}x_1 = \varepsilon^{1/(1-\alpha)}x_1,$$

we obtain from (4), (5)

$$\frac{d^2x_1}{dy^2} = \left(\frac{dx_1}{dy}\right)^{2-\alpha}\left[\psi(y) + \sum_{i>0}\varepsilon^{\beta i}x_1^i\psi_{00i}(y) + \sum_{p,r}{}' \sum_{j\geq 0}\varepsilon^{\beta(j+pk+r)}x_1^j\left(\frac{dx_1}{dy}\right)^{pk+r}\psi_{prj}(y)\right], \tag{6}$$

$$x_1(0) = y_0, \qquad x_1'(0) = 1/C \qquad (\psi(y) = \psi_{000}(y)).$$

We shall seek

$$x_1(y) = x_{1\varepsilon}(y) = x_{00}(y) + \sum_{m+n>0} x_{mn}(y)\varepsilon^{m\beta+nk\beta}. \tag{7}$$

Substituting (7) into (6) and equating terms with the same powers of ε, we obtain a double sequence of equations

$$\frac{d^2x_{00}}{dy^2} - \psi(y)\frac{dx_{00}}{dy} = 0, \qquad x_{00}\big|_{y=y_0} = 0, \qquad x'_{00}\big|_{y=y_0} = \frac{1}{C}; \tag{8}$$

$$\frac{d^2x_{mn}}{dy^2} - (2-\alpha)\psi(y)\left(\frac{dx_{00}}{dy}\right)^{1-\alpha}\frac{dx_{mn}}{dy} = U_{mn}, \qquad x_{mn}\big|_{y=y_0} = x'_{mn}\big|_{y=y_0} = 0, \tag{9}$$

where U_{mn} is expressed in terms of x_{ij}, dx_{ij}/dy, $i \leqslant m$, $j \leqslant n$, $i+j < m+n$:

$$U_{mn} = \sum C_{pru}\psi_{prt}\left(\frac{dx_{00}}{dy}\right)^{2-\alpha+pk+r}\prod_{i=1}^{t} x_{m_in_i}\cdot\prod_{j=1}^{u}\frac{x'_{m_jn_j}}{x'_{00}}. \tag{10}$$

Equation (8) is solvable by quadratures, and from the form of its solution expression (3) for the jump follows. Solving successively equations (9) with respect to x_{mn} for $y_0 \leqslant y < y_0 + B = \bar{y}$, we obtain the following growth estimates for $x_{mn}(y)$ near $\bar{y}$ $(y < \bar{y})$:

$$x_{mn} = O(1)(\bar{y}-y)^{-\frac{(m+1)+nk}{1-\alpha}} = O(1)(\bar{y}-y)^{-(m+1)\beta-nk\beta}, \tag{11}$$

$$\frac{dx_{mn}}{dy} = O(1)(\bar{y}-y)^{-\frac{(m+1)+nk}{1-\alpha}-1} = O(1)(\bar{y}-y)^{-(m+1)\beta-nk\beta-1}.$$

We shall assume that the first zone ends at $y = y_k$, where $y_k < \bar{y}$, $\bar{y}-y_k = O(\varepsilon^{\sigma})$, $\sigma < 1$. To it there corresponds the value $x = x_k = O(\varepsilon^{\gamma_1})$, $\gamma_1 = \dfrac{1-\sigma\alpha}{1-\alpha} > 1$. At the same point we have the value

$$\left.\frac{dy}{dx}\right|_{x=x_k} = p_0 = O\left(\varepsilon^{-\frac{1-\sigma}{1-\alpha}}\right) = O\left(\varepsilon^{-(1-\sigma)\beta}\right).$$

The second zone extends from $x = x_k$ to $x = O(\varepsilon)$. In this zone the jump in y will be small together with ε, while y'_x changes from p_0 to a finite quantity. In the second zone we solve the Cauchy problem for (1) under the conditions:

$$y\Big|_{x=x_k} = y_k, \qquad \frac{dy}{dx}\Big|_{x=x_k} = p_0. \tag{12}$$

In this zone we shall seek an expansion of y (and y'_x) in integral powers of ε:

$$y = y_k + (y - y_k) = y_k + \sum_{i=1}^{\infty} \varepsilon^i y_i = y_k + \varepsilon \sum_{i=1}^{\infty} \varepsilon^i z_i \qquad (\varepsilon z_i = y_i),$$

$$y'_x = p + \sum_{i=1}^{\infty} \varepsilon^i y'_{ix}. \qquad y_i\big|_{x=x_k} = y'_i\big|_{x=x_k} = 0 \left(i > 0;\ \varepsilon z_{n-1} = \int_{x_k}^{x} y'_{n-1}\, dx \right). \tag{13}$$

Hence in this zone we have $(x = x_k + \varepsilon\tau)$

$$\varphi(x, y, y') = \varphi\left(x_k + \varepsilon\tau,\ y_k + \varepsilon \sum_{i=1}^{\infty} \varepsilon^i z_i,\ p + \sum_{i=1}^{\infty} \varepsilon^i y'_i \right) = \tag{14}$$

$$= \varphi(p) + \sum_{r,s,t} \varphi_{rst}(p)\, \varepsilon^r \tau^r \varepsilon^s \left(\sum_{i=1}^{\infty} \varepsilon^i z_i \right)^s \left(\sum_{i=1}^{\infty} \varepsilon^i y'_i \right)^t,$$

where $\varphi(p) = \varphi_{000}(p)$. In doing so we assume

$$\varphi_{rst}(p) = O(1)\varphi(p)p^{-t} = O(1)p^{1+\alpha-t} \tag{15}$$

(cf. the expansion (2)). (In the case where in (2) the sums are finite, (15) follows from (2).) In the first approximation equation (1), by virtue of (13), (14), takes the form

$$\varepsilon \frac{dp}{dx} + \varphi(p) = 0, \qquad p\big|_{x=x_k} = p_0, \tag{16}$$

whence p is found by quadratures. From the same equation it follows that

$$\frac{d}{dx} = -\frac{1}{\varepsilon} \frac{1}{\varphi(p)} \frac{d}{dp}. \tag{17}$$

Substituting formulas (13) into equation (1) and using the expansion (14) and formula (17), we obtain the sequence of equations

$$\varphi(p)\frac{dy_n'}{dp} - \varphi'(p)y_n' = W_n, \qquad y_n'(x_k) = y'\big|_{p=p_0} = 0^*, \tag{18}$$

$$z_n = \frac{1}{\varepsilon}y_n = \frac{1}{\varepsilon}\int_{x_k}^{x} y_n'\,dx = -\int_{p_0}^{p}\frac{y_n'}{\varphi(p)}\,dp, \tag{19}$$

where W_n is expressed in terms of $y_i, z_i,\ i < n$.

The following estimates hold:

$$y_n' = O(1)\,p\,p_0^{n(1-\alpha)}, \qquad z_n = O(1)\,p_0^{n(1-\alpha)}. \tag{20}$$

Under our assumptions and the estimate for p_0, we obtain the asymptotic convergence of (13).

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* Thus, in the second zone, where x and y change infinitesimally little, it turned out to be expedient to use p as the argument, i.e. the first approximation to y_x'.

After a sufficient number of terms has been obtained in the expansions in powers of ε of the functions $x = x_\varepsilon(y)$ or $y = y_\varepsilon(x)$ in each zone, so that the residual in the corresponding equations will be of order $O(\varepsilon^N)$, to estimate the difference R_n between the exact solution of (4) or (1) and the approximation obtained, we apply the method of integral equations. In this way we obtain that R_n in the first zone will be of order $O(\varepsilon^N)$ everywhere, except in a neighborhood of the point y_k, where the estimate for R_n deteriorates somewhat. An analogous assertion holds also in the second zone.

The case when $\alpha = 0$, i.e. $\varphi(x, y, y') = O(1)y'$, includes the quasilinear case studied in (1), and, for the initial value $y'|_{x=0} = C/\varepsilon$, gives a jump by formula (3) with $\alpha = 0$. In this case the width of the second zone will be of order $\varepsilon \ln \varepsilon$.

The case when φ grows as $O(1)|y'|^{1-\delta}$ is studied analogously; for the initial value

$$y'\big|_{x=0} = C/\varepsilon^{\frac{1}{1+\delta}}$$

a jump phenomenon occurs, and (3) is valid with α replaced by $-\delta$. The width of the first two zones will be of order ε^k, where $k < 1$.

The limiting case will be when $\alpha = 1$, i.e. when $\varphi(x, y, y') = O({y'}^2)$. In this case an initial jump is observed if $y'|_{x=0} = e^{C/\varepsilon}$.

After the curve has passed through the first two zones, it enters the third zone with bounded values of y and y'. In this case the asymptotics is constructed in the same way as, for example, in (4).

If, instead of the Cauchy problem, we considered a boundary-value problem, then the picture obtained would be the same as in (1). We would obtain, as in (1), the phenomenon of a jump at one of the ends. The fact that such endpoint jumps are possible for φ growing no faster than ${y'}^2$ is in agreement with the well-known theorem of S. N. Bernstein (5), namely that under precisely these conditions the boundary-value problem for (1) is solvable.

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Note: Figure translations are in progress. See original paper for figures.