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Abstract

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COHOMOLOGY AND DIMENSION OF UNIFORM SPACES

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In the first part of the paper we define the cohomology groups of uniform spaces and proximity spaces and formulate the basic properties of these cohomologies. In the second part the connection is studied between the cohomology groups of a uniform space and its dimension in the sense of Yu. M. Smirnov ([5]) and Isbell ([4]).

§ 1. **Cohomology of uniform spaces.** For a pair of complexes (K, L), $K \supset L$ (generally speaking, infinite), consider the group $C^q(K, L; G)$ of q-dimensional cochains taking only finitely many values in the group G on the simplexes of $K \setminus L$. The coboundary of such a cochain again takes only finitely many values on the simplexes of $K \setminus L$. This makes it possible to define in the usual way the cohomology groups of the cochain complex $\{C^q(K, L; G), \delta\}$, which we call the **finite-valued cohomology groups** of the pair (K, L) and denote by $H^q(K, L; G)$. The coboundary homomorphism $\delta : H^{q-1}(L; G) \to H^q(K, L; G)$ and the homomorphism $f^* : H^q(K', L'; G') \to H^q(K, L; G)$ for a simplicial mapping $f : (K, L) \to (K', L')$ are defined in the usual way.

Let X be a uniform space, and Y its subspace. For each uniform covering α define a pair of complexes (X_α, Y_α) as follows: X_α is the nerve of the covering α, and Y_α is its subcomplex consisting of the simplexes for which the intersection of the elements of the covering corresponding to the vertices has nonempty intersection with Y.

The complex Y_α may also be regarded as the nerve of the covering induced on Y by the covering α. If $\beta > \alpha$ (the covering β is inscribed in the covering α), then any projection
$\omega_\alpha^\beta : (X_\beta, Y_\beta) \to (X_\alpha, Y_\alpha)$ defines a homomorphism
$(\omega_\alpha^\beta)^* : H^q(X_\alpha, Y_\alpha; G) \to H^q(X_\beta, Y_\beta; G)$, which does not depend on the choice of the projection ω_α^β. Thus we obtain a direct spectrum
$\{H^q(X_\alpha, Y_\alpha; G), (\omega_\alpha^\beta)^*\}$, whose limit group will be denoted by $H^q(X, Y; G)$ and called the **cohomology group of the pair of uniform spaces** (X, Y).

A uniform mapping $f : (X, Y) \to (X'Y')$ induces a homomorphism
$f^* : H^q(X', Y'; G) \to H^q(X, Y; G)$. The coboundary operator
$\delta : H^{q-1}(Y; G) \to H^q(Y, Y; G)$ is defined in the usual way. In what follows,

$H^q(X, Y)$ will denote the cohomology group of the pair (X, Y) with coefficient group Z (the group of integers).

It is obvious that the groups $H^q(X, Y; G)$ are invariants of uniform homeomorphisms in both directions. Homological invariants of such mappings, essentially different from ours, were considered earlier by E. S. Tikhomirova ([7]). We also note that finite-valued cohomology of infinite complexes was first introduced by V. Egorov for the classification of uniform mappings of infinite complexes into a sphere.

Theorem 1. The cohomology groups of a pair of uniform spaces have the following properties:

1. If f is the identity mapping, then f^* is the identity homomorphism.
2. $(gf)^* = f^* g^*$.
3. $\delta(f|Y)^* = f^*\delta$, i.e. the diagram

$$\begin{array}{ccc} H^{q+1}(X', Y'; G) & \xrightarrow{f^*} & H^{q+1}(X, Y; G) \\ \uparrow \delta & & \uparrow \delta \\ H^q(Y'; G) & \xrightarrow{(f|Y)^*} & H^q(Y; G) \end{array}$$

is commutative.

4. The sequence

$$\cdots \to H^{q-1}(Y; G) \xrightarrow{\delta} H^q(X, Y; G) \xrightarrow{j^*} H^q(X; G) \xrightarrow{i^*} H^q(Y; G) \to \cdots$$

is exact, where i and j are inclusion mappings, and δ is the coboundary homomorphism.

We shall call uniform mappings $f, g : (X, Y) \to (X', Y')$ **uniformly homotopic** if there exists a uniform mapping

$$F : (X \times I,\ Y \times I) \to (X', Y')$$

such that $F(X, 0) = f(X)$, $F(X, 1) = g(X)$; here $X \times I$ is a uniform space whose uniform structure has as a base the system of coverings that are products of uniform coverings of the space X and finite open coverings of the interval I.

5. If f and g are uniformly homotopic, then $f^* = g^*$.
6. If $X \supset Y \supset Y'$ and the star of the set Y' in some uniform covering of the space X does not intersect $X \setminus Y$, then

   $$i^* : H^q(X, Y; G) \to H^q(X, Y \setminus Y'; G)$$

   is an isomorphism.

The proofs of the assertions of this theorem proceed analogously to the proofs of the properties of ordinary cohomology ([6]).

A triangulation K (in general, infinite) of a uniform space X is called **uniform** if the system of coverings formed by the principal stars of successive barycentric subdivisions of K forms a base of the structure of the space X. A uniform space admitting a uniform triangulation will be called a **uniform polyhedron**.

The underlying space of any complex K may be regarded as a uniform polyhedron; in this case the complex K itself and all its barycentric subdivisions will be uniform triangulations. A mapping of uniform polyhedra which is simplicial for some of their uniform triangulations will be uniform.

Theorem 2. The cohomology of a uniform polyhedron and of any of its uniform triangulations coincide.

Theorem 3 (on simplicial approximation). Let $\bar{K}$ and $\bar{L}$ be uniform polyhedra, taken in some uniform triangulations K and L. If

$$f : \bar{K} \to \bar{L}$$

is a uniform mapping, then there exists a simplicial mapping

$$g : K_n \to L,$$

where K_n is some barycentric subdivision of the complex K, such that g and f are uniformly homotopic.

§ 2. Dimension of uniform spaces

It is said that the dimension ([4]) ΔX of a uniform space X does not exceed n if into every uniform covering one can inscribe a uniform covering of multiplicity $\leq n + 1$. We shall call a uniform mapping f of a uniform space X into the ball R_n **essential** if the mapping f, considered on the inverse image of the boundary sphere S_{n-1} of the ball R_n, cannot be extended to a uniform mapping of the space X into S_{n-1}.

Lemma 1. If $\Delta X = n$, then there exists an essential mapping of the space X into the ball R_n, and every mapping of the space X into a ball of larger dimension is inessential.

The proof of this assertion is based on an analogous theorem of Yu. M. Smirnov ([5]) for proximity spaces and on some results of Isbell ([4]).

Lemma 2 (Borsuk' s lemma). *Let (X, Y) be a pair of uniform spaces and let $f, g : Y \to K$ be uniform mappings of Y into the uniform polyhedron K. Then, if f is uniformly homotopic to g and f can be extended over X to a uniform mapping $F : X \to K$, then g can also be extended to a uniform mapping $G : X \to K$, and the extension G may be chosen uniformly homotopic to the mapping F.*

Lemma 3. *For a uniform cover α of a space X, having a nerve of finite dimension, there exists a uniform mapping $f : X \to X_\alpha$ of the space X into the nerve of the cover α, regarded as a uniform polyhedron, such that the cover by inverse images of the principal stars of the complex X_α is inscribed in the cover α.*

Lemmas 2 and 3 were proved by Isbell ([4]).

For a uniform mapping f of a uniform space X into the oriented sphere S_n, the element f^*Z, where Z is a generating element of the group $H^n(S_n)$, is called the **degree** of the mapping f. We shall call an element $e \in H^n(Y)$ **extendable** if it is the image of some element $\tilde{e} \in H^n(X)$ under the homomorphism $i^* : H^n(X) \to H^n(Y)$.

The proofs of the following generalizations of the known theorems of Hopf and of Theorem 6 are analogous to the proofs that may be found in ([1,3]).

Theorem 4 (on extension of mappings). *Let (X, Y) be a pair of uniform spaces, $\Delta X \leq n + 1$, and let f be a uniform mapping of the space Y into the n-sphere S_n. In order that the mapping f be extendable over X to a uniform mapping, it is necessary and sufficient that the degree e of the mapping be extendable; moreover, if the element $\tilde{e} \in H^n(X)$ is an extension of the element e, then there exists an extension $F : X \to S_n$ of the mapping f with degree $\tilde{e}$.*

Theorem 5 (classification theorem). *The set of classes of uniform mappings of the n-dimensional space X into the sphere S_n, uniformly homotopic to one another, is in one-to-one correspondence with the elements of the group $H^n(X)$.*

Theorem 6. *If $\Delta X \leq n$, then for every subspace Y of the uniform space X one has $H^{n+1}(X, Y) = 0$, and there exists a subspace Y_0 such that $H^n(X, Y_0) \neq 0$.*

Let us make several remarks on the relation of this theory to proximity spaces. It is known that for every proximity space P one can construct a precompact structure consisting of all finite Δ-covers of the space P. The dimension of the uniform space obtained in this way is called by Yu. M. Smirnov ([5]) the dimension of the proximity space. It is natural to call the cohomology groups of this uniform space the cohomology groups of the proximity space P. In this case Theorem 6 gives a cohomological characterization of the dimension of a proximity space.

For an arbitrary coefficient group G, define the cohomological dimension $\Delta_G X$ of a uniform space X as the greatest of the integers such that, for some subspace $Y_0 \subset X$,

$$H^n(X, Y_0; G) \neq 0.$$

For such cohomological dimensions the results of M. F. Bokshtein ([2]) on the comparison of the dimensions of the space X for different coefficient groups remain valid; these results were proved earlier for bicompact spaces.

For this purpose the following algebraic assertions are needed:

Theorem 7. *$C^q(K, L; G) \cong C^q(K, L) \otimes G$ and, consequently, for a uniform space X and an arbitrary coefficient group there exist*

there will be an exact sequence

$$0 \to H^n(X) \otimes G \to H^*(X; G) \to H^{n+1}(X) * G \to 0.$$

(the universal coefficient formula).

Theorem 8. For an exact sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ there exists an exact sequence

$$\cdots \to H^{n-1}(X; G_3) \to H^n(X; G_1) \to H^n(X; G_2) \to H^n(X; G_3) \to \cdots.$$

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References

[1] P. Alexandroff, Proc. Roy. Soc. London, A, **189**, 11 (1947).
[2] M. F. Bokshtein, Tr. Mosk. matem. obshch., **5**, 3 (1956).
[3] W. Hurewicz, H. Wallman, *Dimension Theory*, IL, 1948.
[4] J. R. Isbell, Pacific J. Math., **9**, No. 1, 107 (1959).
[5] Yu. M. Smirnov, Matem. sborn., **38** (80), 3, 283 (1956).
[6] N. Steenrod, S. Eilenberg, *Foundations of Algebraic Topology*, Moscow, 1958.
[7] E. S. Tikhomirova, DAN, **120**, No. 3, 475 (1958).

Note: Figure translations are in progress. See original paper for figures.