A. M. GUTKIN

1960

SovietRxiv

Abstract

Full Text

THEORY OF ELASTICITY

A. M. GUTKIN

FLOW OF A VISCOPLASTIC MEDIUM BETWEEN ROTATING DISKS

(Presented by Academician Yu. N. Rabotnov on 23 V 1960)

A steady flow of a viscoplastic medium between two identical rigid rough parallel disks is considered. It is assumed that the distance between the disks, $2h$, is much smaller than the linear dimensions of the disks.

The disks rotate about a common axis of symmetry with some angular velocity ω. The equilibrium equations, after terms tending to zero have been discarded, are written in the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0, \tag{1}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0. \tag{2}$$

Since the height of the medium located between the disks is much smaller than the radial dimensions, in the first approximation one may take

$$\gamma_{rz} = \frac{\partial v_r}{\partial z}, \tag{3}$$

$$\varepsilon_r = \varepsilon_\theta = \varepsilon_z = 0, \tag{4}$$

$$h = \left|\frac{\partial v_r}{\partial z}\right|. \tag{5}$$

Equations (1) and (2) in this approximation become

$$-\frac{\partial p}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \rho\omega^2 r = 0, \tag{6}$$

$$\frac{\partial p}{\partial z} = 0. \tag{7}$$

Integration of equations (6) and (7) gives

$$\tau_{rz} = \left(\frac{dp}{dr} - \rho\omega^2 r\right) z + C. \tag{8}$$

Since the solution must be symmetric with respect to the plane $z = 0$, the constant C in equation (8) should be set equal to zero. By virtue of the symmetry of the problem, it is sufficient to consider only the region of the medium $z > 0$.

The velocity v_r is obtained from (8) after integration of the relation

$$\tau_{rz} = \mu \frac{\partial v_r}{\partial z} - \tau_0, \tag{9}$$

where μ is the plastic viscosity; τ_0 is the limiting shear stress of the medium. (For definiteness, the special case is considered in which the viscoplastic medium flows out of the gap between the disks.)

Since $v_r = 0$ at $z = h$, we have

$$v_r = \frac{\tau_0}{\mu}(z - h) + \frac{1}{2\mu}\left(\frac{dp}{dr} - \rho\omega^2 r\right)(z^2 - h^2). \tag{10}$$

In the region $z < z_0$, where $|\tau_{rz}| < \tau_0$, the velocity v_r does not depend on z; denote it by v_0. From equations (8) and (10) one obtains

$$z_0 = \frac{\tau_0}{|dp/dr - \rho\omega^2 r|}, \tag{11}$$

$$v_0 = \frac{\tau_0}{\mu}\left[\left(\rho\omega^2 r - \frac{dp}{dr}\right)(h^3 - z_0^3) + \frac{3}{2}\tau_0(z_0^2 - h^2)\right]. \tag{12}$$

Now the flow rate Q can be calculated:

$$Q = 4\pi r v_0 z_0 + 4\pi r \int_{z_0}^{h} v_r\, dz = \frac{4\pi r}{3\mu}\left[\left(\rho\omega^2 r - \frac{dp}{dr}\right)(h^3 - z_0^3) + \frac{3}{2}\tau_0(z_0^2 - h^2)\right]. \tag{13}$$

After substituting into equation (13) its expression from relation (11), one obtains

$$Q = \frac{4\pi r}{3\mu}\left[\left(\rho\omega^2 r - \frac{dp}{dr}\right)h^3 - \frac{3}{2}\tau_0 h^2 + \frac{\tau_0}{2(\rho\omega^2 r - dp/dr)^2}\right]. \tag{14}$$

Equation (14) can be used to calculate the derivative dp/dr. The following special cases are of interest.

A. **Solid disks of radius R approach each other with a small velocity U.** Between the flow rate in each cylindrical section of the medium and the relative velocity U there is the relation

$$Q = \pi r^2 U. \tag{15}$$

Then, if in equation (14) the third term on its right-hand side is neglected, which is small in practically important cases of developed flow, we obtain

$$\frac{dp}{dr} = \left(\rho\omega^2 - \frac{3\mu U}{4h^3}\right) r - \frac{3\tau_0}{2h}. \tag{16}$$

If the pressure in the medium at the points where $r = R$ is taken to be zero, then integration of equation (16) gives

$$p = \left(\frac{3\mu U}{8h^3} - \frac{\rho\omega^2}{2}\right)(R^2 - r^2) + \frac{3\tau_0}{2h}(R - r). \tag{17}$$

The forces F acting on each of the rotating plates are equal to

$$F = 2\pi \int_0^R rp\, dr = \frac{\pi R^4}{4}\left(\frac{3\mu U}{4h^3} - \rho\omega^2\right) + \frac{\pi R^3 \tau_0}{h}. \tag{18}$$

From equation (18) one obtains

$$U = \frac{16Fh^3}{3\pi\mu R^4} + \frac{4\rho\omega^2 h^3}{3\mu} - \frac{16\tau_0 h^2}{3\mu R}. \tag{19}$$

From formula (19) one can calculate the rate of approach of the disks for various values of the compressive force F and the angular velocity ω. The formula is valid both for $F \leq 0$ and for $\omega = 0$, subject to the condition, however, that the velocity U be greater than zero.

B. A viscoplastic medium continuously enters the space between the disks. Let the distance between the disks be constant, but let there be, at the center of one of them, a circular opening of radius a, through which the viscoplastic medium flows into the space between the disks. (The case in which the distance between the disks changes simultaneously with this is considered analogously to what is set forth below.) If the last term on the right-hand side of equation (14) is omitted and the equation is solved with respect to dp/dr, we obtain:

$$\frac{dp}{dr} = \rho\omega^2 r - \frac{3\tau_0}{2h} - \frac{3\mu Q}{4\pi r h^3}. \tag{20}$$

If the distance between the disks is constant, then the flow rate Q does not depend on the coordinate r, and after integration of equation (20) one obtains

$$p_1 - p_2 = \frac{\rho\omega^2}{2}(a^2 - R^2) + \frac{3\mu Q}{4\pi h^3}\ln\frac{R}{a} + \frac{3\tau_0}{2h}(R - a), \tag{21}$$

where p_1 is the pressure at $r = a$ and p_2 is the pressure at $r = R$.

If the angular velocity ω is equal to zero, formula (21) becomes the equation

$$\Delta p = \frac{3\mu U}{4\pi h^3}\ln\frac{R}{a} + \frac{3\tau_0}{2h}(R - a). \tag{22}$$

Relation (22) almost coincides with the equation obtained earlier by N. V. Tyabin ([1]), who investigated the last of the particular cases of a viscoplastic medium considered by us. The difference in the equations is explained by an error made by N. V. Tyabin at the end of his work.

Formulas (19) and (21) can be used both for the experimental determination of the constants of a viscoplastic medium and for calculating the flow rate of consistent lubricants in labyrinth seals.

Moscow Power Engineering Institute

Received
17 V 1960

REFERENCES

[1] N. V. Tyabin, *DAN*, **96**, No. 1 (1954).

Note: Figure translations are in progress. See original paper for figures.