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Abstract

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ON ONE METHOD FOR FINDING ALL SOLUTIONS OF NONLINEAR OPERATOR EQUATIONS

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At the present time there is a substantial difference between methods of approximate solution of nonlinear equations in finite-dimensional spaces and nonlinear operator equations. In finite-dimensional spaces, for example for algebraic equations with one unknown, there exist well-developed methods for finding **all** solutions of a given equation, whereas for operator equations all approximate methods known up to now work only under the assumption of uniqueness of the solution.

In the present paper we aim to fill this gap partially and to set forth the idea of a method for finding all solutions for operator equations of the form

$$y = F(y)$$

where $F(y)$ is some completely continuous operator acting from a Banach space B into the same space. Let D_F^0 be a convex compact subset of the space B, mapped into the compact set $D_{F_1} \subset D_F^0$ by means of the operator F. Since the abstract function $F(y)$ is uniformly continuous on D_F^0, for each $\gamma > 0$ one can find an $\varepsilon(\gamma) \leqslant \gamma$ such that from the condition $\rho(y_1, y_2) \leqslant \varepsilon(\gamma)$ it follows that $\rho(F(y_1), F(y_2)) \leqslant \gamma$.

Now let $N_{\varepsilon(\gamma)}$ be a finite $\varepsilon(\gamma)$-net in D_F^0; then this net, under the operator F, will pass into a γ-net in $D_F^{(1)}$. Let x_0 be a fixed point, i.e. $x_0 = F(x_0)$. Denote by Λ_γ the $\varepsilon(\gamma)$-star about the point x_0, i.e. the set of those points of the net $N_{\varepsilon(\gamma)}$ whose distance from x_0 is less than or equal to $\varepsilon(\gamma)$. Let $\widetilde{\Lambda}_\gamma = F(\Lambda_\gamma)$; let y_0 belong to $\Lambda_{\varepsilon(\gamma)}$ and let $\tilde{y}_0$ be its image: $\tilde{y}_0 = F(y_0)$.

The points y_0 and $\tilde{y}_0$ are points of D_F^0; let us estimate their distance. We have:

$$\rho(\tilde{y}_0, y_0) \leqslant \rho(\tilde{y}_0, x_0) + \rho(x_0, y_0) \leqslant \gamma + \varepsilon(\gamma) \leqslant 2\gamma.$$

Let the nets $\varepsilon(1), \varepsilon\left(\frac{1}{2}\right), \dots, \varepsilon\left(\frac{1}{2^n}\right) = \varepsilon_n, \dots$ be constructed in D_F^0. If a fixed point x_0 exists, then one can find a point y_{x_0} of the $\varepsilon(\gamma)$-net in D_F^0 such that

$\rho(y_{x_0}, F(y_{x_0})) \leqslant 2\gamma$. Consequently, to each fixed point x_0 one can assign a sequence of points $\{y_{x_0}^{(n)}\}$ such that $y_{x_0}^{(n)}$ belongs to the $\varepsilon(\frac{1}{2^n})$-net in D_F^0 and, moreover,

$$\rho(y_{x_0}^{(n)}, F(y_{x_0}^{(n)})) \leqslant \frac{1}{2^{n-1}}.$$

Let an operator equation $y = F(y)$ be given, for which it is not known in advance how many fixed points of $F(y)$ exist and where they are located. We find an $\varepsilon(\frac{1}{2^n})$-finite net in D_F^0 and the corresponding

a $\frac{1}{2^n}$-net in D_F; let these be N_n and $\widetilde{N}_n$,

$$\widetilde{N}_n = F(N_n).$$

We consider the pairs

$$\tilde{y}_1^{(n)} = F(y_1^{(n)}), \dots, \tilde{y}_s^{(n)} = F(y_s^{(n)}),$$

where $y_i^{(n)}$ are vertices of the net N_n, and select those pairs for which

$$\rho(y_i^{(n)}, \tilde{y}_i^{(n)}) \leq \frac{1}{2^{n-1}};$$

if there are no such pairs, then fixed points do not exist.

We arrange the pairs marked at each step (for each n) into chains according to the following rule. Suppose that at the n-th step the pairs found are

$$\Delta_1^{(n)}, \dots, \Delta_{s_n}^{(n)},$$

and at the $(n+1)$-st step the pairs are

$$\Delta_1^{(n+1)}, \dots, \Delta_{s_{n+1}}^{(n+1)}.$$

Then we shall regard a pair $\Delta_k^{(n)}$ as corresponding to a pair for which

$$1) \quad \rho(y_k^{(n)}, y_j^{(n+1)}) \leq 2\varepsilon_n,$$
$$2) \quad \rho(\tilde{y}_k^{(n)}, \tilde{y}_j^{(n+1)}) \leq \frac{1}{2^{n-1}};$$

if a fixed point is present, such pairs will certainly be found (see Fig. 1).

At each step we must compute a finite number of values of the function $F(y)$ and carry out a finite number of comparisons; since, in checking whether corresponding pairs exist at the n-th and $(n+1)$-st steps, we are dealing with inequalities, the computation of the values $F(y)$ may also be performed approximately.

Thus, for each fixed point we shall find a sequence of pairs, and to different fixed points there correspond, as is easy to see, different sequences of corresponding pairs. Let us make the following observation. Call a 2γ-star of some net a set of vertices of this net whose pairwise distances are not greater than 2γ. If there is a fixed point, then all vertices of such a star will simultaneously enter into the marked pairs, and for further investigation it is sufficient to restrict ourselves to considering one of these pairs. Moreover, after the "corresponding pairs" have been detected at the $(n-1)$-st and n-th steps, the net should be constructed and the values $F(y)$ computed only at those vertices of the $\varepsilon(\frac{1}{2^{n+1}})$-net which are at a distance from the vertices of the $\varepsilon(\frac{1}{2n})$-net entering into the corresponding pairs no greater than

$$\frac{1}{2^{\,n-1}}.$$

Even if the number of solutions in the domain under study is infinite, at each finite step the process requires only a finite number of operations and may be regarded as a process of root separation.

Naturally, one should take up the calculation of the number of necessary operations and the most economical methods of constructing nets; these questions will be illuminated in the next paper.

Fig. 1

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Note: Figure translations are in progress. See original paper for figures.