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Abstract

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Conjugate Systems of the 3rd Order and the Problem of Their Focal Transformations

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1. Common to all known generalizations of a conjugate net is the fact that each of them is based on one or another definition of conjugacy of two directions; for this reason all these generalizations may be called conjugate nets and systems of the 2nd order. Analogously, one may speak of conjugate nets and systems of the 3rd order, i.e., of those based on the conjugacy of three directions (which in essence means the study of the properties of a surface with respect to the third differential neighborhood of a point). The author' s paper (^{3}) is the first attempt to consider such systems. In the present paper we consider holonomic conjugate systems of the 3rd order and solve the problem of their focal transformations in the three-dimensional case.

2. Conjugate nets and systems of the 3rd order are defined as follows (^{3}):

Definition 1. Three tangent directions $d_1,\ d_2,\ d_3$ of a p-dimensional surface V_p $(p \geqslant 3)$ have conjugacy of the 3rd order if they make equal to zero all trilinear forms associated with the cubic asymptotic forms of the surface:

$$\Phi^{\alpha_3}(d_1 d_2 d_3) = 0.$$

Definition 2. Three families of lines on a p-dimensional surface V_p $(p \geqslant 3)$ have conjugacy of the 3rd order if at each point of the surface the tangents to them form a triple of conjugate directions.

Definition 3. A p-conjugate system of the 3rd order is such a p-dimensional surface V_p $(p \geqslant 3)$ on which there exists a nonsingular net consisting of p families of lines, every triple of which is conjugate.

3. In n-dimensional projective space we consider a p-dimensional surface V_p, to each point A of which there is attached a projective frame of $n+1$ base points $A,\ A_{\alpha_1},\ A_{\alpha_2},\ A_{\alpha_3},\ A_\sigma$, where the points $A,\ A_{\alpha_1}$ $(\alpha_1, i,\ j,\ k = 1, \dots, p)$ form a basis of the first osculating space (the tangent space), the points $A,\ A_{\alpha_1},\ A_{\alpha_2}$ form a basis of the second osculating space, and the points $A,\ A_{\alpha_1},\ A_{\alpha_2},\ A_{\alpha_3}$ form a basis of the third osculating space of the surface.

Along with the indices α_2 and α_3, we shall also use multi-index symbols: (ii), (ij) instead of α_2 and (iii), (iij), (ijk) instead of α_3, which are convenient in the case when the quadratic Φ^{α_2} and cubic Φ^{α_3} asymptotic forms of the surface are reduced to canonical (simultaneous) form (3).

The geometric meaning of conjugacy of the 3rd order can be expressed by the following theorem:

Theorem. *If on a p-dimensional surface V_p ($p \geqslant 3$), with quadratic asymptotic forms Φ^{α_2} reduced to canonical form*

the edge point $A_{(ij)}$ does not leave the 2nd osculating space when the point A of the surface is displaced in the direction AA_k, then the three directions AA_i, AA_j, AA_k, tangent to the base lines $\omega^i, \omega^j, \omega^k$, have conjugacy of the 3rd order.

Below we shall speak of conjugate systems only of three-dimensional ones ($p = 3$) and of the 3rd order, without emphasizing this each time.

4. A three-dimensional conjugate system of the 3rd order, existing with an arbitrariness of 12 functions of 3 arguments (3), is not holonomic—it is not stratified into one-parameter families of two-dimensional surfaces along the lines of the conjugate net. The minimal additional condition that must be adjoined in order for the stratification to take place is the condition of symmetry of the coefficient a^i_{jk} in the indices j, k in the general expansion of the form $\omega^i_j = a^i_{ij}\omega^i + a^i_{jj}\omega^j + a^i_{jk}\omega^k$, i.e., the condition $a^i_{jk} = a^i_{kj}$. Such a holonomic system we shall call a **primitive** holonomic conjugate system. A primitive holonomic conjugate system exists with an arbitrariness of 9 functions of 3 arguments, just as does its special case for $\omega^i_j = 0$, corresponding to the immobility of the conjugate net (3).
5. We shall arrive at a number of new holonomic conjugate systems with the aid of the following geometric constructions:

Definition 1. By the first holonomic conjugate system we shall mean a three-dimensional surface such that, after reducing the quadratic asymptotic forms Φ^{α_2} to canonical form, in each of the three-dimensional spaces $(AA_iA_jA_{(ij)})$ there exists at least one point P which does not leave the four-dimensional space $(AA_1A_2A_3A_{(ij)})$ when the point A of the surface is displaced in the direction AA_k.

Theorem 1. *The first holonomic conjugate system is a conjugate system of the 3rd order with restrictions imposed on it in the form of the relations*

$$[\omega^{(ij)}_{(ij)}\omega^i\omega^j] = 0, \qquad [\omega^{(ij)}_{(jk)}\omega^j\omega^k] = 0 \qquad (i, j, k = 1, 2, 3).$$

The locus of the points P mentioned in the definition forms a line.

Definition 2. By the second holonomic conjugate system we shall mean a three-dimensional surface such that, after reducing the quadratic asymptotic forms Φ^{α_2} to canonical form, in each of the four-dimensional spaces $(AA_1A_2A_3A_{(ij)})$ there exists at least one point P which does not leave this space when the point A of the surface is displaced in the direction AA_k.

Theorem 2. *The second holonomic conjugate system is a conjugate system of the 3rd order with restrictions imposed on it in the form of the relations* $[\omega_{(ij)}^{(it)}\omega^i\omega^j] = 0$. *The locus of the points P mentioned in the definition forms a line.*

Existence theorem. *The 1st and 2nd holonomic conjugate systems exist with an arbitrariness of 6 functions of 3 arguments. They are stratified into one-parameter families of two-dimensional surfaces along the lines of the conjugate net.*

6. The family of lines $AA_{(12)}$, attached to each point A of the three-dimensional surface, forms a complex. When the point A moves in the direction AA_3 (along the line $\omega^1 = \omega^2 = 0$, $\omega^3 \neq 0$), from this complex there is distinguished a ruled (in general non-developable) surface.

Definition. By the principal holonomic conjugate system we shall mean a three-dimensional surface with a fixed net of base lines $\omega^1, \omega^2, \omega^3$ (the condition of immobility $\omega_j^i = 0$), for which, after reducing the quadratic asymptotic forms Φ^{α_2} to canonical form, from each

of the complex of lines $AA_{(ij)}$, under motion in the direction AA_k, a developable surface is singled out.

Remark. The stratifiability of the principal holonomic system is a consequence of the equality $\omega_j^i = 0$.

Theorem. *The principal holonomic conjugate system is a conjugate system of the 3rd order with a conjugate net of base lines $\omega^1, \omega^2, \omega^3$, as referred to in the definition, with restrictions imposed on it in the form of the relations*

$$[\omega_{(ij)}^{(ii)}\omega^i\omega^j] = 0, \qquad [\omega_{(jk)}^{(ii)}\omega^j\omega^k] = 0, \qquad [\omega_{(jk)}^{(ij)}\omega^j\omega^k] = 0,$$

$$[\omega_{(ij)}^{i}\omega^i\omega^j] = 0, \qquad [\omega_{(jk)}^{i}\omega^j\omega^k] = 0,$$

where the last relation is an additional one, ensuring the coincidence of the point $A_{(ij)}$ with the focus of the ray $AA_{(ij)}$.

Proof. We seek such a point P (the focus) of the ray of the complex $AA_{(12)}$ which would move along this ray when $\omega^1 = \omega^2 = 0$, $\omega^3 \neq 0$. Let $P = \xi A + A_{(12)}$; then

$$dP = (d\xi + \xi\omega_0^0 + \omega_{(12)}^0)A + (\xi\omega^1 + \omega_{(12)}^1)A_1 + (\xi\omega^2 + \omega_{(12)}^2)A_2$$
$$+ (\xi\omega^3 + \omega_{(12)}^3)A_3 + \omega_{(12)}^{(11)}A_{(11)} + \omega_{(12)}^{(22)}A_{(22)} + \omega_{(12)}^{(33)}A_{(33)} + \omega_{(12)}^{(12)}A_{(12)}$$
$$+ \omega_{(12)}^{(13)}A_{(13)} + \omega_{(12)}^{(23)}A_{(23)} + \sum_{\alpha_3} \omega_{(12)}^{\alpha_3} A_{\alpha_3}.$$

For $\omega^1 = \omega^2 = 0,\ \omega^3 \neq 0$, we obtain

$$\widetilde{dP} + (d\xi + \xi\tilde{\omega}_0^0 + \tilde{\omega}_{(12)}^0)A + \tilde{\omega}_{(12)}^1 A_1 + \tilde{\omega}_{(12)}^2 A_2 + (\xi\omega^3 + \tilde{\omega}_{(12)}^3)A_3 +$$
$$+ \tilde{\omega}_{(12)}^{(11)}A_{(11)} + \tilde{\omega}_{(12)}^{(22)}A_{(22)} + \tilde{\omega}_{(12)}^{(33)}A_{(33)} + \tilde{\omega}_{(12)}^{(12)}A_{(12)} + \tilde{\omega}_{(12)}^{(13)}A_{(13)} + \tilde{\omega}_{(12)}^{(23)}A_{(23)} + \sum_{\alpha_3} \tilde{\omega}_{(12)}^{\alpha_3} A_{\alpha_3}.$$

On the other hand, by condition,

$$\widetilde{dP} = \theta A + \theta^{(12)} A_{(12)}.$$

Comparing the two expressions for $\widetilde{dP}$, we obtain relations from which the assertions of the theorem follow.

Existence theorem. *The principal holonomic conjugate system exists with arbitrariness of 3 functions of 3 arguments.*

To the Pfaff system for the three-dimensional conjugate system (3) we add the relations of the preceding theorem and the equation $\omega_j^i = 0$. Such an augmented system is in involution. The arbitrariness of the solution is determined by the last character

$$s_3 = 3\binom{1,2}{}$$

7. The proof of the existence theorem for the principal holonomic conjugate system is at the same time also a proof of the existence of focal transformations of this system, carried out by "mixed two-index" points $A_{(12)}, A_{(13)}, A_{(23)}$ from the 2nd osculating space. But these transformations are far from unique. The point is that, as a result of the canonization of the quadratic asymptotic forms Φ^{α_2}, the position of each of the three points $A_{(ij)}$ is determined only up to its location in the corresponding four-dimensional space $(AA_1A_2A_3A_{(ij)})$, which is determined uniquely. Therefore, in each of the three directions we shall have a four-parameter family of focal transformations.

Theorem. *The principal holonomic conjugate system admits 3 four-parameter families of focal transformations, carried out by "mixed two-index" points* $A_{(ij)}$*. All focal surfaces are conjugate systems of the 3rd order.*

Proof. It remains for us to prove only the last assertion. Consider the focal surface (P), described by the focus

$P \equiv A_{(12)}$, and bring its quadratic asymptotic forms $\widetilde{\Phi}^{\alpha_2}$ to canonical form. Choose the base points of the frame (P) as follows:

$$P \equiv A_{(12)}, \qquad P_1 \equiv A_{(112)}, \qquad P_2 \equiv A_{(122)}, \qquad P_3 \equiv A, \qquad P_{(13)} \equiv A_1,$$

$$P_{(23)} \equiv A_2, \qquad P_{(33)} \equiv A_3.$$

It can be shown that this can always be done without disturbing the canonization of the asymptotic forms. We take the remaining base points of the frame (P) in any positions admissible for them.

Let us prove that $dP_{(13)}$ does not leave the second osculating space Π_2 of the surface (P) when $\omega^1 = \omega^3 = 0,\ \omega^2 \neq 0$. Indeed,

$$dP_{(13)} \equiv dA_1 = \omega_1^0 A + \omega_1^1 A_1 + \omega_1^2 A_2 + \omega_1^3 A_3 + \omega^1 A_{(11)} + \omega^2 A_{(12)} + \omega^3 A_{(13)}.$$

For $\omega^1 = \omega^3 = 0,\ \omega^2 \neq 0$ and $\omega_j^i = 0$, we obtain

$$\tilde{d}P_{(13)} = \tilde{\omega}_1^0 P_3 + \tilde{\omega}_1^1 P_{(13)} + \omega^2 P \in \Pi_2,$$

whence the assertion of the theorem follows.

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CITED LITERATURE

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[3] G. I. Trushin, DAN, 129, No. 1 (1959).

Note: Figure translations are in progress. See original paper for figures.