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Abstract

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SOLUTION OF A MIXED PROBLEM FOR A PARABOLIC SYSTEM BY THE METHOD OF POTENTIALS

(Presented by Academician I. G. Petrovskii, 12 I 1960)

A parabolic, in the sense of Petrovskii, system of differential equations of order $2p$ is considered:

$$L(x,t,\partial/\partial t,\partial/\partial x)u \equiv (\partial/\partial t - A(x,t,\partial/\partial x))u = f(x,t); \qquad (1)$$

where

$$x = (x_1, \dots, x_n), \qquad u(x,t) = (u_1(x,t), \dots, u_N(x,t)), \qquad f(x,t) = (f_1, \dots, f_N),$$

$$A\left(x,t,\frac{\partial}{\partial x}\right) \equiv \sum_{k_1+\dots+k_n=2p} A_{k_1,\dots,k_n}(x,t)\frac{\partial^{2p}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}} +$$

$$+ \sum_{k_1+\dots+k_n<2p} A_{k_1,\dots,k_n}(x,t)\frac{\partial^{k_1+\dots+k_n}}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}} \equiv A_0(x,t,\partial/\partial x) + A_1(x,t,\partial/\partial x).$$

Here $A_{k_1,\dots,k_n}(x,t)$, $k_1+\dots+k_n \le 2p$, $k_i \ge 0$, are square matrices of order N with sufficiently smooth elements in $\overline{\Omega}$ (Ω is a domain in the space $t, x_1, \dots, x_n$, bounded below and above respectively by the planes $t=0$ and $t=T>0$, and laterally by a sufficiently smooth surface Γ, for example satisfying the Lyapunov conditions, whose normal nowhere for $0 \le t \le T$ is parallel to the axis Ot).

Let λ be the roots of the determinant

$$\det\|\lambda E - A_0(x,t,i\alpha)\| = 0 \qquad (2)$$

for real $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_1^2+\dots+\alpha_n^2 = 1$, satisfying the parabolicity condition

$$\operatorname{Re}\lambda_k(x,t,i\alpha) < -\delta(x,t), \qquad k=1,\dots,N, \tag{3}$$

where $\delta(x,t) \geq \delta > 0$ for all $(x,t) \in \overline{\Omega}$.

We seek a solution of the mixed problem for equation (1) satisfying the conditions

$$u(x,t)\big|_{t=0} = \varphi(x), \tag{4}$$

$$\partial^i u(x,t)/\partial n^i\big|_{\Gamma} = \psi_i(\Gamma), \qquad i=0,\dots,p-1, \tag{5}$$

where n is the direction of the inward normal to the surface Γ. The consideration of boundary conditions more general than (5), containing independent linear combinations of the unknown functions and their derivatives with respect to the normal up to order $(p-1)$, differs from the case considered here only by the choice of other potentials instead of (6) and by more cumbersome calculations.

It is assumed that the functions $f(x,t)$, $\varphi(x)$, $\psi_i(\Gamma)$, $i=0,\dots,p-1$, satisfy certain compatibility conditions at the points of the set $\Gamma \cap (t=0)$; therefore conditions (4), (5) can be replaced by homogeneous ones.

conditions (4^0), (5^0), and the functions $f(x,t)$ in $(1')$, by new functions satisfying the conditions

$$\left.\frac{\partial^i f(x,t)}{\partial n^i}\right|_{\Gamma\cap(t=0)} = 0, \qquad i=0,\dots,p-1.$$

Moreover, Levi' s method ([1,2]) makes it possible, as will be seen from what follows, to restrict oneself to considering the problem (4^0), (5^0) for a system with constant coefficients containing only the highest derivatives with respect to x:

$$L_0(\partial/\partial t, \partial/\partial x)u \equiv (\partial/\partial t - A_0(\partial/\partial x))u = f(x,t). \tag{1}$$

In this note the Green matrix $G(x,\xi;t,\tau)$ will be constructed for problem (1), (4^0), (5^0) (i.e., the solution of the homogeneous problem (1), (4^0), (5^0), regular everywhere in Ω except for the point $(\xi_1,\dots,\xi_n,\tau)$, and having at this point a singularity of the same type as the fundamental solution $U(x,\xi;t,\tau)$ ([3])):

$$G(x,\xi;t,\tau) = U(x,\xi;t,\tau) + G_0(x,\xi;t,\tau),$$

where $G_0(x,\xi;t,\tau)$ is a solution regular in Ω of (1^0) (system (1), where $f(x,t) \equiv 0$), satisfying the boundary conditions (5), with

$$\psi_i(\Gamma) = -\left.\frac{\partial^i U(x,\xi;t,\tau)}{\partial n_x^i}\right|_{\Gamma} = 0, \qquad i=0,\dots,p-1$$

(for $t < \tau$, $U(x,\xi;t,\tau)$ is assumed to be equal to zero). The Green function by other methods for the heat equation was first constructed in ([4]), and for a parabolic system in the plane in ([5]).

In ([6]) it is proved that $U(x,\xi;t,\tau)$ is an entire analytic function of the variables x and ξ of order $2p/(2p-1)$; therefore, to construct $G(x,\xi;t,\tau)$ it is enough to solve problem $(1^0), (4^0), (5^0)$, where $\psi_i(\Gamma)$ are entire analytic functions of order $2p/(2p-1)$ in x at each point of Γ, and continuously differentiable functions with respect to t.

By a potential of order k for equation (1^0), with continuous density $\mu_k(\tau,\xi_1,\dots,\xi_n)$, $0 \le k \le 2p-1$, distributed on Γ, we shall mean the matrix

$$V_k(x,t) = \int_{\Gamma_t} \frac{\partial^k U(x,\xi;t,\tau)}{\partial n_\xi^k}\, \mu_k(\xi,\tau)\, d\Gamma_t, \qquad \Gamma_t = \Gamma \cap (0 \le \tau \le t). \tag{6}$$

It is obvious that in Ω the matrix $V_k(x,t)$, $0 \le k \le 2p-1$, is a regular solution of (1^0).

We shall seek the solution of problem $(1^0), (4^0), (5^0)$ in the form

$$u(x,t) = \sum_{k=0}^{p-1} V_k(x,t). \tag{7}$$

In this case, for the p unknown vector densities $\mu_k(\xi,\tau)$, by virtue of $(5')$, one obtains p vector equations

$$\psi_i(x,t)|_\Gamma = \sum_{k=0}^{p-1} \frac{\partial^i V_k(x,t)}{\partial n_x^i}\bigg|_\Gamma, \qquad i = 0,\dots,p-1 \tag{8}$$

(the condition (4^0) is fulfilled automatically if the $\mu_k(\xi,\tau)$ that will be obtained from (8) turn out to be integrable over Γ).

To simplify the exposition, let us suppose that Ω is a cylinder,

$$\Omega = D \times [0,T],$$

where D is a bounded domain in the plane $t=0$, $D - \overline{D} = C$ is the boundary of D, $\Gamma = C \times [0,T]$, $\Gamma_t = C \times [0,t]$, $\Gamma_{\tau t} = C \times [\tau,t]$. Denote by l_x the tangent plane in the plane $t=0$ to the surface C, passing through the point $x \in C$, and by $L_{x,\tau}$ the part of the tangent plane to Γ with direction l_x and generatrix parallel to Ot, lying between the planes $t=0$ and $t=\tau \le T$ (in the case of a noncylindrical surface Γ, instead of the plane $L_{x,\tau}$ one must take the plane tangent to Γ at the point (x,t)). We also introduce the tangent potentials, for $0 \le k \le 2p-1$,

$$\widetilde{V}_k^{(x_0)}(x,t) = \int_0^t d\tau \int_{l_{x_0}} \frac{\partial^k U(x,\xi;t,\tau)}{\partial n_{x_0}^k}\, \tilde{\mu}_k(\xi,\tau)\, d\xi = \int_{L_{x_0,t}} \frac{\partial^k U}{\partial n_{x_0}^k}\, \mu_k(\xi,\tau)\, d\xi\, d\tau, \tag{9}$$

where the function $\tilde{\mu}_k(\xi,\tau)$ is constructed from $\mu_k(\xi,\tau)$ as follows: a sufficiently small $\varepsilon > 0$ is chosen, and on the set $L_{x_0,T} \cap ((|\xi - x_0| \le \varepsilon) \times [0,T])$ the value $\tilde{\mu}_k(\xi,\tau)$ is taken to be equal to the value of μ_k at that point of Γ whose normal intersects $L_{x_0,T}$ at the point (ξ,τ); at all other points of $L_{x_0,T}$, $\tilde{\mu}_k(\xi,\tau)$ is defined so that it has the same smoothness as $\mu_k(\xi,\tau)$ and tends to zero as $|\xi| \to \infty$, uniformly with respect to $\tau \in [0,T]$. In view of the assumed smoothness of Γ, $\varepsilon > 0$ may be taken fixed for the whole surface C.

Lemma 1. The difference

$$D_t^{s/2p}\left(\frac{\partial^i V_k(x,t)}{\partial n_x^i} - \frac{\partial^i \widetilde{V}_k^{(x_0)}(x,t)}{\partial n_x^i}\right) = \int_{\Gamma_t \cup L_{x_0,t}} W_{s,k,i}^{(x_0)}(x,\xi;t,\tau)\,\nu_k(\xi,\tau)\,d\xi\,d\tau, \tag{10}$$

$$0 \le s,k,i \le 2p-1; \qquad (x,t) \in \Gamma_t \cup L_{x_0,t}; \qquad x_0 \text{ is an arbitrary point on } C;$$

$\nu_k(\xi,\tau) = \mu_k(\xi,\tau)$ for $(\xi,\tau) \in \Gamma_t$ and $\nu_k(\xi,\tau) = \tilde{\mu}_k(\xi,\tau)$ for $(\xi,\tau) \in L_{x_0,T}$, is, uniformly with respect to x_0, a completely continuous operator in the space of continuous functions $\mu_k(\xi,\tau)$ ($\tilde{\mu}_k(\xi,\tau)$), and

$$\left\|W_{s,k,i}^{(x_0)}(x,\xi;t,\tau)\right\| \le$$

$$\le \frac{A_{s,k,i}}{(t-\tau)^{\frac{k+i+s+1-\gamma}{2p}}} \exp\left\{-\eta_0\,|x-\xi|^{2p/(2p-1)}/(t-\tau)^{1/(2p-1)}\right\}, \tag{11}$$

where γ is the Lyapunov constant of the surface C; $A_{s,k,i}$ and η_0 are constants. $D_t^{s/2p}$ is the differentiation operator in t of order $s/2p$ ([7,8]).

Lemma 2 (fundamental). The equality

$$D_t^{\left(1-\frac{k+i+1}{2p}\right)} \frac{\partial^i \widetilde{V}_k^{(x_0)}(x,t)}{\partial n_{x_0}^i}\bigg|_{x=x_0} =$$

$$= B(x_0)\,\mu_k(x_0,t) + \int_{L_{x_0,t}} \overline{W}_{k,i}(x_0,\xi;t,\tau)\,\tilde{\mu}_k(\xi,\tau)\,d\xi\,d\tau =$$

$$= B_{ik}(x_0)\,\mu_k(x_0,t) + \sum_{s=1}^{n} \int_{L_{x_0,t}} \overline{\overline{W}}_{s;k,i}(x_0,\xi;t,\tau)\frac{\partial \tilde{\mu}_k(\xi,\tau)}{\partial \xi_s}\,d\xi\,d\tau, \tag{12}$$

holds, where

$$B_{ik}(x) = \frac{\pi(-1)^{i+k}}{\Gamma\left(\frac{i+k+1}{2p}\right)\sin\left(\frac{i+k+1}{2p}\pi\right)} \int \cdots \int_{(n-1)} \frac{\partial^{i+k} U(\xi', 1)}{\partial n_x^{i+k}}\, d\xi', \tag{13}$$

$$\xi' = (\xi_1, \dots, \xi_{n-1}, 0),$$

$$\frac{\partial^{i+k} U(\xi', 1)}{\partial n_x^{i+k}} = \frac{1}{(2\pi)^n} \int \cdots \int_n e^{i(\xi',\alpha)} (n_x, \alpha)^{i+k} e^{-A_0(\alpha)}\, d\alpha_1 \cdots d\alpha_n, \tag{14}$$

$$|\det B_{ik}(x)| \geq \delta > 0 \qquad \text{for all } x \in C, \tag{15}$$

and $\overline{W}$ and $\overline{\overline{W}}$ have estimates of the form (11), if in them one sets $k+i+s=2p$, $\gamma=1$, and, respectively, $k+i+s=2p-1$, $\gamma=1$. The $B_{ik}(x)$ are computed explicitly if (1) is a single equation.

Applying the operator $D^{1/2p}$ to the $(p-1)$-st equation of the system (8), by Lemmas 1 and 2 we obtain an integral equation of the second kind with respect to

of the vector $\mu_{p-1}(x,t)$, from which, according to (15), $\mu_{p-1}(x,t)$ is determined by integrals of $\mu_k(x,t)$, $0 \leq k \leq p-1$, with kernels of type $\widetilde{\overline{W}}_{k,i}$. Substituting $\mu_{p-1}(x,t)$ in the $(p-2)$-nd equation (8) and applying to it the operator $D^{2/2p}$, we find, as above, $\mu_{p-2}(x,t)$ (the legitimacy of these operations is proved), and so on. As a result, instead of (8) we obtain the system of equations

$$\mu_s(x,t) - \sum_{r=0}^{p-1} \sum_{k=1}^{n} \int_{\Gamma_t} \widetilde{W}_{s,r}^{(k)}(x,t;\xi,\tau) \frac{\partial \mu_r(\xi,\tau)}{\partial \xi_k}\, d\xi\, d\tau = \widetilde{\psi}_s(x,t) \tag{8'}$$

with matrices $\widetilde{W}_{s,r}^{(k)}$ of type $\widetilde{\overline{W}}$; $\widetilde{\psi}_s(x,t)$ are entire analytic functions of x of order $2p/(2p-1)$. System $(8')$ is solved by the method of successive approximations:

$$\mu_s(x,t) = \mu_s^{(0)} + \mu_s^{(1)} + \cdots + \mu_s^{(m)} + \cdots,$$

where $\mu_s^{(m)}$ is determined from $\mu_s^{(m-1)}$, $m \geq 1$, in the usual way. If

$$M_k^{(s)}(t) = \max \left| \frac{\partial^s \mu_r(\xi,\tau)}{\partial \xi_1^{s_1} \cdots \partial \xi_n^{s_n}} \right| \quad \text{for } 0 \leq r \leq p-1, \quad (\xi,\tau) \in \Gamma_t,$$

then one can prove that

$$M_k^{(0)}(t) \leq \frac{(C_0 t^{1/2p})^k}{\Gamma(k/2p)} M_0^{(k)}(t) \leq (C_1 t^{1/2p})^k, \tag{16}$$

i.e., the series for $\mu_s(x,t)$ converge for $|t| \leq (1/C_1)^{2p}$ (inequality (16) is written by virtue of known estimates of the Taylor coefficients for entire functions of order $2p/(2p-1)$ [9]). Since C_1 depends only on the boundary C, by means of several steps one can reach $t = T$.

This proves:

Theorem. *If the boundary of the domain* Γ*, whose normal is nowhere parallel to the t-axis, satisfies the Lyapunov conditions and there exists a fundamental solution of (1)* [2]*, then there exists the Green matrix* G *of problem (1), (4), (5).*

With the aid of the Green matrix, the solution of problem (1), (4), (5) is written in the usual way as an integral over Ω.

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References

[1] E. E. Levi, UMN, vol. 8 (1941).
[2] S. D. Eidelman, Mat. sbornik, 38, no. 1 (1956).
[3] I. G. Petrovskii, Bull. Moscow State Univ., sect. A, no. 7 (1939).
[4] A. N. Tikhonov, Bull. Moscow State Univ., sect. A, no. 8 (1939).
[5] V. P. Mikhailov, DAN, 126, no. 6 (1959).
[6] I. M. Gelfand, G. E. Shilov, *Generalized Functions*, vol. 2 (1958).
[7] A. Zygmund, *Trigonometric Series*, Moscow-Leningrad, 1939.
[8] V. P. Mikhailov, DAN, 129, no. 6 (1959).
[9] A. I. Markushevich, *Analytic Functions*, 1950.