APPROXIMATION OF FUNCTIONS WITH PRESERVATION OF HOMOGENEOUS BOUNDARY CONDITIONS

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Abstract

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MATHEMATICS

A. L. FUKSMAN

APPROXIMATION OF FUNCTIONS WITH PRESERVATION OF HOMOGENEOUS BOUNDARY CONDITIONS

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1. Let D be an m-dimensional domain with boundary Γ, and let the function u have continuous partial derivatives in $\overline{D}$ up to some order and satisfy the boundary condition

$$u = \frac{\partial u}{\partial n} = \ldots = \left.\frac{\partial^{s-1} u}{\partial n^{s-1}}\right|_{\Gamma} = 0. \tag{1}$$

In the present note we consider the approximation of such functions by expressions of the form

$$\varphi(x_1, \ldots, x_m) P_n(x_1, \ldots, x_m),$$

where $P_n(x_1, \ldots, x_m)$ is a polynomial of degree not exceeding n in each of the arguments, and $\varphi(x_1, \ldots, x_m)$ is a fixed function satisfying condition (1); an estimate is given for the rate of approximation of the function and its derivatives as $n \to \infty$, depending on the smoothness of u.

This problem was considered earlier in the case where Γ is a sufficiently smooth boundary, for $s = 1$ (^{1}) and $s = 2$ (^{2}). We obtain the same estimates for arbitrary s in the case where Γ belongs to a certain class of surfaces composed of sufficiently smooth pieces.

2. Let x, t be points of m-dimensional space, $x = (x_1, \ldots, x_m)$; Q_α is the m-dimensional cube, $-\alpha \leq x_i \leq \alpha$, $1 \leq i \leq m$; D^k is any partial derivative of order k; $C^k(D)$ is the space of functions having continuous derivatives up to order k in $\overline{D}$; $C^{k,1}(D)$ is the space of functions $f \in C^k(D)$ for which $D^k f$ satisfies $\mathrm{Lip}_m\, 1$; $\omega_{C(D)}(f; \varepsilon)$ is the modulus of continuity of a function $f \in C(D)$, $\omega^{(k)}_{C(D)}(f; \varepsilon) = \max_{0 \leq l \leq k} \omega_{C(D)}(D^l f; \varepsilon)$; $\widetilde{C}^k$ is the space of 2π-periodic functions $f \in C^k(Q_\pi)$; by analogy $\widetilde{C}^{k,1}$, $\omega^{(k)}_{\widetilde{C}}(f; \varepsilon)$ are defined; Γ^s is the set of functions equal to zero on Γ together with derivatives up to

order $s-1$; if $\psi = \psi_1\psi_2 \dots \psi_p$, then we put

$$\varphi^{[\lambda]}_{i_1 \dots i_\lambda} = \varphi[\psi_{i_1}\psi_{i_2} \dots \psi_{i_\lambda}]^{-1}, \quad 1 \leq i_1 < i_2 < \dots < i_\lambda \leq p;$$

$\varphi^{[\lambda]}$ denotes any of the functions $\varphi^{[\lambda]}_{i_1 \dots i_\lambda}$; $\varphi^{[0]} = \varphi$.

We introduce the class of boundaries $\Gamma(k,d)$, subjecting it to the following condition: $\Gamma \in \Gamma(k,d)$ if to each point $x \in \Gamma$ one can assign in this manner a neighborhood Ω_x and a mapping $\sigma_x(\overline{x}_i = \overline{x}_i(t_1, \dots, t_m)$, $1 \leq i \leq m$, $t \in \Omega_x)$ of this neighborhood onto the cube Q_1, such that $\overline{x}_i \in C^{k,1}(\Omega_x)$, the Jacobian of σ_x at the point x is not zero, $\Omega_x \cap D$ passes into the part $\Delta_{q(x)}$ of the cube Q_1, determined by the inequalities $0 < \overline{x}_1 \leq 1$, $0 < \overline{x}_2 \leq 1, \dots, 0 < \overline{x}_{q(x)} \leq 1$, where $q(x) \leq d \leq m$. Roughly speaking, $q(x)$ denotes the number of $(m-1)$-dimensional curved "faces" meeting at the point $x \in \Gamma$.

Denote by Σ_i the hyperplane $\overline{x}_i = 0$, and by Γ_i denote $\sigma_x^{-1}(\Sigma_i \cap Q_1)$. We define the class $\Phi(k,d,s)$ of admissible functions φ by the conditions:
1) $\varphi \in C^{k,1}(D)$;
2) $\varphi(x) > 0$, $x \in D$;
3) for $x \in \Gamma$ there exists such a spherical neighborhood $O_x \subset \Omega_x$ that in $D \cap O_x$, φ is representable in the form

$$\varphi = (\varphi_1\varphi_2 \dots \varphi_{q(x)})^s,$$

moreover: a) $\varphi_i \in C^{k,1}(O_x)$; b) if $t \in O_x - \Gamma_i$, then $\varphi_i(t) \neq 0$; c) if $t \in \Gamma_i$, then $\varphi_i(t) = 0$, but $|\operatorname{grad} \varphi_i(t)| \neq 0$.

3. The main result is contained in the following theorem.

Theorem. *Let the domain* D *have boundary* $\Gamma \in \Gamma(k,d)$*, and let a function* $\varphi \in \Phi(k,d,s)$ *be defined in it. If* $u \in C^k(D)$*,* $u \in \overset{0}{\Gamma}{}^s$*,* $k \geq ds$*, then there exists a sequence of polynomials* $P_n(x)$ *of degree not exceeding* n *in each of the variables, for which*

$$\|u - \varphi P_n\|_{C^l(D)} \leq M n^{-(k-l)} \left[\omega(n^{-1}) + n^{-1}\|u\|_{C^k(D)}\right],$$

where $l = 0, 1, \dots, k$*;* $\omega(\varepsilon) = \omega^{(k)}_{C(D)}(u; \varepsilon)$*, and* M *depends only on* D, φ, k*.*

We outline the proof. Without loss of generality, one may assume that $D \subset Q_\alpha$, $0 < \alpha < 1$, and all neighborhoods O_x assigned to $x \in \Gamma$, in view of $\varphi \in \Phi(k,d,s)$, also lie in Q_α. Further, one may suppose that $u \equiv 0$ outside $O'_{x_0} \subset O_{x_0}$, $x_0 \in \Gamma$; the general case is reduced to this by decomposing u into the sum of a finite number of functions $(^8)$. The proof is based on the following lemmas.

Lemma 1 (basic). *Let* $\varphi = \psi_1\psi_2 \dots \psi_p$*;* $\psi_i \in \widetilde{C}^{k,i}$ $(1 \leq i \leq p)$*;* $\varphi f \in \widetilde{C}^k$*;* $f \in \widetilde{C}^{k-p}$*,* $k \geq p$*;* $\varphi^{[\lambda]} f \in \widetilde{C}^{k-\lambda}$ $(0 \leq \lambda \leq p)$*, and* $\|\varphi^{[\lambda]} f\|_{\widetilde{C}^{k-\lambda}} \leq N(f)$*;*

$$\omega^{(k-\lambda)}_{\widetilde{C}}(\varphi^{[\lambda]} f; \varepsilon) \leq \omega_f(\varepsilon).$$

Then there exists a sequence of trigonometric polynomials $\widetilde{P}_n$ of order not exceeding n in each of the variables, for which

$$\|\varphi(f - \widetilde{P}_n)\|_{\widetilde{C}^l} \leq A_1 n^{-(k-l)}[\omega_f(n^{-1}) + N(f)n^{-1}], \qquad 0 \leq l \leq k,$$

where A_1 does not depend on f or n. If f is even, then $\widetilde{P}_n$ contains only cosines.

The polynomials $\widetilde{P}_n$ are constructed from the function f as in (^{1}), with the sole difference that, for the basic operator, not Jackson' s kernel is taken, but a somewhat modified one. The estimate is obtained from a certain identity which expresses $\varphi(f - \widetilde{P}_n)$ in terms of the functions $\varphi^{[\lambda]} f$.

Lemma 2. *Under the assumptions of the theorem, u can be extended to O_x so that*

$$\bar{u} \in C^k(O_x); \qquad \|\bar{u}\|_{C^k(O_x)} \leq A_2 \|u\|_{C^k(D)}; \qquad \omega^{(k)}_{C(O_x)}(\bar{u}; \varepsilon) \leq A_2 \omega(\varepsilon);$$

$$D^\alpha \bar{u}(t) = 0, \quad \text{if } t \in \Gamma_i, \quad 1 \leq i \leq q(x), \quad 0 \leq \alpha \leq s-1, \tag{3}$$

where A_2 does not depend on u, ε. If moreover $u \equiv 0$ outside $O'_x \subset O_x$, then $\bar{u} \equiv 0$ outside O_x.

The proof is obtained by using Hestenes' method.

Since $\varphi \in \Phi(k, d, s)$ in O_{x_0}, the function $\varphi = (\varphi_1 \ldots \varphi_q)^s$; we enumerate the sq factors composing it with one index, putting $\psi_l = \varphi_j$, $j = \left[\dfrac{l-1}{s}\right] + 1$, so that $\varphi = \psi_1 \psi_2 \ldots \psi_p$, where $p = sq \leq sd$. Introduce also the function $v = \bar{u}\varphi^{-1}$. Using (3), one can prove that $\varphi^{[\lambda]}_{i_1 \ldots i_\lambda} v = \varphi[\psi_{i_1} \ldots \psi_{i_\lambda}]^{-1} v = u[\psi_{i_1} \ldots \psi_{i_\lambda}]^{-1} \in C^{k-\lambda}(O_{x_0})$, where $0 \leq \lambda \leq p$, with the corresponding estimate of the modulus of continuity. Extend ψ_j $(1 \leq j \leq p)$ to Q_1, preserving smoothness, so that $\psi_j \equiv 0$ outside Q_α, and v may be regarded as already extended, since $v \equiv 0$ outside $O_{x_0} \subset Q_\alpha$. Then $\varphi^{[\lambda]} v \in C^{k-\lambda}(Q_1)$;

$$\|\varphi^{[\lambda]} v\|_{C^{k-\lambda}(Q_1)} \leq A_3 \|u\|_{C^k(D)}; \qquad \omega^{(k)}_{C(Q_1)}(\varphi^{[\lambda]} v; \varepsilon) \leq A_3 \omega(\varepsilon).$$

Applying the usual device, to a function $f \in C(Q_1)$ one may associate an even function $\tilde{f}$: $\tilde{f}(x) = f(\cos x_1, \ldots, \cos x_m)$. If $f \equiv 0$ outside Q_α, then

$$\|f\|_{C^k(Q_1)} \leq A_4 \|\tilde{f}\|_{\widetilde{C}^k}, \qquad \omega^{(k)}_{\widetilde{C}}(\tilde{f}; \varepsilon) \leq A_5 \omega^{(k)}_{C(Q_1)}(f; \varepsilon),$$

where A_4 and A_5 depend only on α and k. Therefore, for the function $f = \tilde{v}$ the conditions of Lemma 2 are fulfilled, so that the sequence of trigonometric polynomials indicated in it can be found. In view of the evenness of $\tilde{v}$, these polynomials can be written in the form

$$\tilde{P}_n = \sum_{i_1,\ldots,i_m=0}^{n} a^{(n)}_{i_1\ldots i_m} (\cos x_1)^{i_1} \cdots (\cos x_m)^{i_m}.$$

Then the polynomials

$$P_n = \sum_{i_1,\ldots,i_m=0}^{n} a^{(n)}_{i_1\ldots i_m} x_1^{i_1} \cdots x_m^{i_m}$$

are the desired ones. Indeed,

$$\begin{aligned}\|u - \varphi P_n\|_{C^l(D)} \le \|\bar{u} - \bar{\varphi} P_n\|_{C^l(Q_1)} = \|\varphi(v - P_n)\|_{C^l(Q_1)} \le A_4 \|\tilde{\varphi}(\tilde{v} - \tilde{P}_n)\|_{C^l} \\ \le A_4 A_1 n^{-(k-l)} [\omega_{\tilde{v}}(n^{-1}) + n^{-1} N(\tilde{v})] \le \\ \le A_4 A_1 A_5 A_3 A_2 n^{-(k-l)} [\omega(n^{-1}) + n^{-1} \|u\|_{C^k(D)}],\end{aligned}$$

whence (2) follows.

4. Remark. The theorem remains valid if, instead of $k \ge sd$, one requires only $k \ge s - 1$ for $s > 1$ and $k \ge 1$ for $s = 1$. In the case $k = 0$, $s = 1$ the theorem is true if $\Gamma \in \Gamma(1, d)$, $\varphi \in \Phi(1, d, 1)$. The proof in this case is somewhat more complicated; the decisive role is still played by the identity mentioned in Lemma 1, but the differential properties of $v = u\varphi^{-1}$ have to be studied in greater detail.

If $D \in Q_\pi$, then the theorem is valid if the P_n are regarded as trigonometric polynomials.

Let the plane domain D be 2π-periodic in ξ and bounded in η, i.e., from $(\xi; \eta) \in D$ it follows that $(\xi + 2\pi; \eta) \in D$, $|\eta| < a$. For such domains the theorem is also valid if φ and u are 2π-periodic in ξ, and

$$P_n = \varphi(\xi, \eta) \sum_{i,k=0}^{n} \eta^i (a_{i,k} \cos k\xi + b_{ik} \sin k\xi).$$

An analogous assertion is also valid for $m > 2$, and the periodicity may be in several variables.

In the case $m = 2$, in order that $\Gamma \in \Gamma(k, d)$, it is necessary and sufficient that Γ be piecewise smooth, with each smooth piece having, in local coordinates, an equation $\eta = f(\xi)$, where $f \in C^{k,1}$, and that the angle α between two adjacent pieces, directed into the domain, lie between 0 and π. If $\alpha > \pi$, then for $s = 1$ one can find such k, l and a function $u \in C^k(D),\ u \in \overset{\circ}{\Gamma}{}^s$, that (2) is not satisfied, whatever φ and P_n may be. The same can be asserted also for some domains having angle $\alpha = \pi$.

In the case $m > 2$, the condition $\Gamma \in \Gamma(k, d)$ also imposes a restriction on the number $q(x)$ of "faces"of the boundary that meet at the point $x \in \Gamma$. Apparently this restriction can be removed. The entire difficulty here is contained in the proof of Lemma 2.

The results presented can be applied to estimate the rate of convergence of approximate solutions of partial differential equations obtained by variational methods.

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Rostov-on-DonState University

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REFERENCES

1. I. Yu. Kharrik, *Matem. sborn.*, **37** (79), 353 (1955).
2. I. Yu. Kharrik, *Matem. sborn.*, **47** (89), 177 (1959).
3. G. M. Fichtenholz, *Course of Differential and Integral Calculus*, 1, Moscow-Leningrad, 1958.

Note: Figure translations are in progress. See original paper for figures.