MATHEMATICS

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Abstract

Full Text

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A LINEAR PARTIALLY ORDERED TOPOLOGICAL SPACE

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1. Convergence in a partially ordered space is usually introduced by the (exact) bound of a countable bounded set ([1]), or by a Banach norm ([1–3]). Here a space is constructed with a partial order, a topology, and preservation of the usual connections between relations of inequality and convergence, except for the requirement that a bound exist for an infinite bounded set. The most natural way of constructing such a space turns out to be a direct generalization of a partially ordered Banach space ([3]). In the space obtained, methods are constructed for a two-sided estimate of the solution of a nonlinear functional equation. Mainly, methods are considered under conditions different from the known ones.

2. Let X be a linear lattice ([3]) and a linear topological space ([4]) with a basis of neighborhoods G of zero θ. We prescribe the neighborhoods to be balanced ([4]), i.e. $\lambda G \subset G$ for $|\lambda| \leqslant 1$.

Definition 1. We shall call a neighborhood G **monotone** if from the relations $|x| \leqslant |y|$, $y \in G$, it follows that $x \in G$. If every neighborhood is monotone, then the basis will be called **monotone**.

Definition 2. A linear lattice which is a linear topological space with a monotone basis will be called a **linear partially ordered topological space** (KT-**space**).

If the topology is introduced by a Banach norm, then the notions of a KT-space and of a partially ordered Banach space coincide.

Notation: $\sup\{x, y\} = x \vee y$, $\inf\{x, y\} = x \wedge y$ ([1]). Sometimes we shall indicate in which space a neighborhood, a bound, and a modulus are taken: G_x, etc.

Let us establish the connection between a KT- and a K-space ([1]) (which is a special case of a linear lattice).

Definition 3. A subset K of a linear lattice X, consisting of certain $x > \theta$, will be called an **open positive** subset if K, together with any pair of its elements u, v, also contains $u \wedge v$ and $\frac{1}{n}u$ (n a natural number).

Theorem 1. If in a K-space X there exists an open positive subset K, then, defining G by the inequality $|x| < u \in K$, we turn X into a KT-space. In this case topological convergence will imply (o)-convergence ([1]). If from the relations $x_n > x_{n+1}$, $\inf\{x_n\} = \theta$, it follows that $x_n < u$ for $n = n(u)$, then from (o)-convergence there follows topological convergence of a (countable) sequence.

Thus, when all the conditions of the theorem are fulfilled, the notions of KT- and K-space coincide. The topology introduced here in a K-space is, in general, different from the topology introduced in ([1]).

Theorem 2. In a KT-space: 1) majorized sequences converge; 2) the modulus, bound, and inequality are continuous; 3) inter-

a piecewise variable has a limit; 4) a monotone variable is comparable with its limit, i.e., if $x_n \to x,\ y_n \to y$, then

1) from $|z_n| \leq |x_n|,\ x = \theta$ it follows that $z_n \to \theta$;
2) $|x_n| \to |x|,\ x_n \vee y_n \to x \vee y$; from $x_n \leq y_n$ it follows that $x \leq y$;
3) from $x_n \leq z_n \leq y_n,\ x = y$ it follows that $z_n \to x$;
4) from $x_n > x_{n+1}$ it follows that $x_n > x$.

The continuity of x_+ follows from the equality $x_+ = \frac{1}{2}(x+|x|)$ and the continuity of the modulus. The continuity of the inequality is established more simply than in ([3]).

Definition 4. A closed linear set $Y \subset X$ will be called a KT-subspace of the subspace X, if the bounds in X and Y coincide, i.e., if for $x, y \in Y$, $(x \vee y)_X$ is $(x \vee y)_Y$.

Y itself is a KT-space; the monotonicity of the bounds follows from the fact that $|x|_Y = |x|_X,\ G_Y = G_X \cap Y$. If X is countably complete ([4]), i.e., the completeness condition is fulfilled for countable sequences, then Y is the same.

Definition 5. KT-spaces X and Z will be called KT-equivalent if a one-to-one correspondence $x = \varphi z$ is established between them, where φ and φ^{-1} are additive, continuous, and positive ([1]).

The transformation φ preserves bounds: $\varphi(x \vee y) = \varphi x \vee \varphi y$. From KT-equivalence follows the topological equivalence ([4]) of X and Z. If X and Z are K-spaces, then φ is their isomorphism ([1]).

3. In a KT-space, theorem 4 ([5]) is valid, describing a method of two-sided approximations to the solution of the equation $P(x) = \theta$, under the additional condition: the process

$$x_{n+1} = U(x_n) \equiv x_n - \Gamma^{-1} P(x_n) \tag{1}$$

converges in the given topology to the solution.

If for some considerations it is known that the initial approximation is comparable with the solution x^* of the equation $x = U(x)$, then the iterative algorithm can be simplified and replaced by a process that gives approximations, improving at each step of the process, comparable with the solution.

Theorem 3. Suppose U is monotonically increasing ([1]), $x^* < x_0$ (or $> x_0$). Construct elements x_k satisfying the inequalities $x_n > x_{n+1} \geq x_n \wedge U(x_n)$ (or $x_n < x_{n+1} \leq x_n \vee U(x_n)$). Then $x_k > x^*$ (or $x_k < x^*$).

Arbitrariness in the choice of x_{n+1} makes it possible to simplify its nature and thereby facilitate the process.

4. In a KT-space the result of § 2 ([6]) is preserved, where a method of alternating upper and lower approximations is constructed. Such alternation makes it possible to halve the computations of the "fork"of approximations. The convergence of the iterative process is additionally required.

Notation: $\bigvee$ denotes one of the meanings $\leq, \geq$.

Theorem 4. Suppose $x_0 < \bar{x}_0$, $\Gamma \Delta x \leq P(x + \Delta x) - P(x)$ for $\Delta x > \theta$, Γ is additive, Γ^{-1} positive, $\Gamma(\bar{x}_0 - x_0) \geq P(\bar{x}_0) \vee -P(x_0)$, and the process (1) converges to the solution x^* of the equation $P(\bar{x}) = \theta$. Define x_k by the algorithm $x_{n+1} = x_n - \Gamma^{-1} z$, where x_0 is one of $x_0, \bar{x}_0$; $\Gamma(x_n - x_{n-1}) \bigvee z_n \bigvee p(x_n)$ $(n = 0, \dots \infty)$; if $x_0 = \underline{x}_0 (= \bar{x}_0)$, then $x_{-1} = \bar{x}_0 (= \underline{x}_0)$. Then

$$x_k \bigvee x^* \bigvee x_{k+1}. \tag{2}$$

Of course, we choose $z_n \neq \Gamma(x_n - x_{n-1})$.

The conditions imposed on the initial approximations can be removed if it is known in advance that they are comparable with the solution x^* of the equation $x = U(x)$.

Theorem 5. Let U decrease monotonically, $x^* < x_0$ (or $> x_0$). Construct x_k satisfying one of the pairs of inequalities

$$x_n \wedge U(x_{n+1}) \leq x_{n+2} < x, \qquad x_n \vee U(x_{n+1}) \geq x_{n+2} > x_n, \qquad x_1 = U(x_0).$$

Then (2) holds.

5. In a KT-space the method of majorants can be refined ((1), p. 466) by constructing a two-sided estimate instead of an estimate in the abstract norm.

Let Y be a KT-subspace of X; let the operators V, W map Y into itself, and let their domain of definition be Y.

Definition 6. We shall call V, W, respectively, a **left** and a **right majorant** if

$$V(y+\Delta y)-V(y)\leq U(x+\Delta x)-U(x)\leq W(z+\Delta z)-W(z)$$

when $y\leq x\leq z,\ \Delta y\leq \Delta x\leq \Delta z$.

Let Y be KT-equivalent to the space $\overline{Y}$: $\varphi Y=\overline{Y}$. Define $\overline{V}=\varphi V\varphi^{-1}$, $\overline{W}=\varphi W\varphi^{-1}$. The corresponding elements will be denoted by $\overline{y}_n=\varphi y_n$, etc.

Theorem 6. If X is countably complete; U is continuous; V,W are left and right majorants; $y_0\leq x_0\leq z_0$; $y_0-V(y_0)\leq x_0-U(x_0)\leq z_0-W(z_0)$; the sequences $\overline{y}_{n+1}=\overline{V}(\overline{y}_n)$, $\overline{z}_{n+1}=\overline{W}(\overline{z}_n)$ $(n=0,\dots,\infty)$ converge to $\overline{y}^*,\overline{z}^*$, then the process $x_{n+1}=U(x_n)$ converges to the solution x^* with rate

$$y^*-y_n\leq x^*-x_n\leq z^*-z_n.$$

Thus,

$$x_0-u_0+y^*\leq x^*\leq x_0-z_0+z^*.$$

If several approximations have been computed, then, obviously,

$$y^*+\sup\{x_n-y_n\}\leq x^*\leq z^*+\inf\{x_n-z_n\}.$$

The choice of the spaces $Y,\overline{Y}$ may, for example, be carried out in the same way as in (7).

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Note: Figure translations are in progress. See original paper for figures.