MATHEMATICS

Academician **L. S. Pontryagin** and **L. V. Rodygin

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Abstract

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APPROXIMATE SOLUTION OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE DERIVATIVES*

1. We shall study the behavior, on a finite time interval, of solutions of the system of differential equations

$$\varepsilon \frac{dx}{dt} = f(x,y), \qquad \frac{dy}{dt} = g(x,y), \tag{1}$$

where $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_l)$ are vectors, and $\varepsilon > 0$ is a small parameter. The functions $f(x,y) = (f_1(x,y), \dots, f_k(x,y))$, $g(x,y) = (g_1, \dots, g_l)$ are assumed to be twice continuously differentiable in a certain (open) domain Γ of the space of variables (x,y), whose form will be specified below. The projection of the domain Γ onto the space of variables y is denoted by G.

It is assumed that for any $y \in G$ the system of fast motions

$$\frac{d\tilde{x}}{d\tau} = f(\tilde{x}, y) \qquad (y = \text{const} - \text{parameter}) \tag{2}$$

has, in the domain under consideration, exactly one rough stable limit cycle $x^*(\tau, y)$. This means that $x^*(\tau, y)$ is a periodic solution of system (2), whose period we shall denote by $T(y)$, and that the multipliers of the system of equations in variations

$$\frac{d\xi_i}{d\tau} = \sum_{j=1}^{k} \frac{\partial f_i[x^*(\tau,y), y]}{\partial x_j} \xi_j \qquad (i = 1, \dots, k) \tag{3}$$

are, in modulus, less than one except for one; these multipliers we denote by $\lambda_1, \dots, \lambda_{k-1}$. It follows from this that $x^*(\tau, y)$ has a domain of attraction $F(y)$; any solution of system (2) with initial value from $F(y)$ tends as $t \to \infty$ to the limit cycle x^*. As is easily seen,

$$\bigcup_{y \in G} F(y) \times y$$

is a domain in the space (x, y), which we identify with our original domain Γ. We shall assume that there exist T_1, T_2 for which $0 < T_1 \le T(y) \le T_2$. Generally speaking, as y approaches the boundary of the domain G, $T(y)$ could tend to 0 or to ∞, but for our further purposes it is always possible to pass from G to some compact subdomain $D \subset G$. Under the assumptions made, $T(y)$ turns out to be a smooth function of y.

Let us introduce into consideration the "averaged system"

$$\frac{d\bar{y}}{dt} = \bar{g}(\bar{y}) = \frac{1}{T(\bar{y})} \int_0^{T(\bar{y})} g(x^*(\tau, \bar{y}), \bar{y})\, d\tau = \int_0^1 g(X(\varphi, \bar{y}), \bar{y})\, d\varphi, \tag{4}$$

* The principal results of the present work were reported at the All-Union Mathematical Congress ([1]).

where it is set* that $X(\varphi, y) = x^*(T(y)\varphi, y)$. We denote by $\{x(t,\varepsilon), y(t,\varepsilon)\}$ the solution of system (1) with initial value $(x_0, y_0) \in \Gamma$, $x_0 \in F(y_0)$, by $\tilde{x}(\tau)$ the solution of system (2) with initial value $\tilde{x}(0) = x_0$ and with parameter $y = y_0$, and by $\overline{y}(t)$ the solution of system (4) with initial value $\overline{y}(0) = y_0$. It is assumed that $\overline{y}(t) \in G$ for $0 \le t \le L < \infty$. By D is denoted a compact subdomain of the domain G, containing $\overline{y}(t)$ $(0 \le t \le L)$ strictly in its interior. Everywhere below it is assumed that $y \in D$.

2. We pass in system (1) to the "fast time" $\tau = t/\varepsilon$:

$$\frac{dx}{d\tau} = f(x, y), \qquad \frac{dy}{d\tau} = \varepsilon g(x, y) \tag{5}$$

and compare (5) with (2). From the theorem on continuous dependence of solutions on a parameter it follows directly that, on any time interval $0 \le t \le \varepsilon t_1$ of length of order ε, $|x(t,\varepsilon) - \tilde{x}(t/\varepsilon)|$, $|y(t,\varepsilon) - y_0|$ can be made arbitrarily small, provided only that ε is sufficiently small. Thus, over a time interval of order ε, the solution of system (1) enters a small neighborhood of the closed curve $C = C_{y_0}$, where C_y denotes $\{x = X(\varphi, y),\ 0 \le \varphi \le 1\} \times y$.

3. Let us trace the further behavior of $\{x(t,\varepsilon), y(t,\varepsilon)\}$. It is natural to expect that $y(t,\varepsilon)$ will be close to the averaged solution $\overline{y}(t)$, while $x(t,\varepsilon)$ will remain near the cycles $C_{\overline{y}(t)}$, performing rapid oscillations along them with period close to $\varepsilon T(\overline{y}(t))$. The precise formulation is as follows:

Theorem. There exists a function $\varphi(t,\varepsilon)$ ("phase"), depending smoothly on t, such that if $\delta > 0$, then for $\delta \le t \le L$

$$\left| \varepsilon \frac{d\varphi}{dt} - \frac{1}{T(\overline{y}(t))} \right| = O(\varepsilon), \tag{6a}$$

$$|x(t,\varepsilon) - X(\varphi(t,\varepsilon), \overline{y}(t))| = O(\varepsilon), \tag{6b}$$

$$|y(t,\varepsilon) - \overline{y}(t)| = O(\varepsilon) \tag{6c}$$

uniformly in t.

4. To study system (1) near C_y, we use a nonsingular transformation of the form

$$\xi = \frac{\partial x^*}{\partial \tau} u_0 + A\left(\frac{\tau}{T(y)}, y\right) u$$

(u_0 is a scalar, u is a vector with $k-1$ components, $A(\varphi, y)$ is a matrix with k rows and $k-1$ columns, having period 1 in φ). With the aid of such a transformation, system (3) can be reduced to the form

$$\frac{du_0}{d\tau} = 0, \qquad \frac{du}{d\tau} = H\left(\frac{\tau}{T(y)}, y\right) u, \tag{7}$$

where $H(\varphi, y) = H(\varphi+1, y)$ is a square matrix of order $k-1$, and the multipliers of the system $\dot{u} = Hu$ are our $\lambda_1, \dots, \lambda_{k-1}$. Hence there follows the existence of a Lyapunov function $W\left(\frac{\tau}{T(y)}, u, y\right)$, a quadratic function of u with period 1 in the first argument, whose total derivative with respect to τ, by virtue of system (7), is

$$\left[\frac{dW}{d\tau}\right]_{(7)} < -\alpha W \qquad (\alpha > 0). \tag{8}$$

* As the initial value $X(0, y)$, any point of the periodic solution of system (2) may be taken, as long as the dependence on y is smooth.

Moreover, one can ensure that the coefficients W are smooth functions of their arguments, so that

$$\left|\frac{\partial W}{\partial \varphi}\right|, \quad \left|\frac{\partial W}{\partial y_i}\right| < \beta_1 W, \quad \left|\frac{\partial W}{\partial u_i}\right| < \beta_2 \sqrt{W},$$

$$\beta_3 \sqrt{W} < |u| < \beta_4 \sqrt{W} \quad (\beta_3 > 0), \tag{9}$$

and the β_i, like α in (8), can be chosen the same for all $y \in D$.

By the change of variables $x = X(\varphi, y) + A(\varphi, y)u$, system (1) is reduced to the form

$$\varepsilon\frac{d\varphi}{dt} = \frac{1}{T(y)} + O(\varepsilon + |u|),$$

$$\varepsilon\frac{du}{dt} = H(\varphi, y)u + O(\varepsilon + |u^2|), \tag{10}$$

$$\varepsilon\frac{dy}{dt} = g[X(\varphi, y), y] + O(\varepsilon + |u|).$$

Computing the total derivative of the function $W(\varphi, u, y)$ with respect to t, by virtue of system (10), one can show that there exist $\varepsilon_1, C_1, C_2 > 0$ such that, for $\varepsilon < \varepsilon_1$, $C_1\varepsilon^2 \leq W \leq C_2$,

$$\varepsilon\left[\frac{dW}{dt}\right]_{(10)} < -\alpha_1 W \quad (\alpha_1 > 0).$$

It follows that the surfaces $W = C$ $(C_1\varepsilon^2 \leq C \leq C_2)$ are contact-free surfaces for system (10), and that if a solution of this system reaches, at time $t = t_{1\varepsilon}$, the surface $W = C_2$, then, as long as it lies in the region $C_1\varepsilon^2 \leq W \leq C_2$, the inequality

$$W(\varphi(t,\varepsilon), u(t,\varepsilon), y(t,\varepsilon)) \leq$$

$$\leq W(\varphi(t_{1\varepsilon},\varepsilon), u(t_{1\varepsilon},\varepsilon), y(t_{1\varepsilon},\varepsilon))e^{-\frac{1}{\varepsilon}\alpha_1(t-t_{1\varepsilon})} = C_2 e^{-\frac{1}{\varepsilon}\alpha(t-t_{1\varepsilon})},$$

holds; hence after a time $t_{2\varepsilon} = O(\varepsilon \ln \varepsilon)$, W becomes $C_1\varepsilon^2$. The solution thus enters inside the surface $W = C_1\varepsilon^2$ and can no longer leave the interior of this surface as long as $y(t,\varepsilon) \in D$. If the solution of system (2) enters, in time t_1, the interior of the surface $W = \frac{1}{2}C_2$, then, for sufficiently small ε, the solution of system (5) will certainly enter the interior of the surface $W = C_2$ in time t_1, so that $t_{1\varepsilon} < \varepsilon t_1$. Using (9), we now obtain that, for $\varepsilon t_1 \leq t \leq t_{2\varepsilon}$, one has

$$|u| < C_3 \exp\{-\alpha_1(t - \varepsilon t_1)/2\varepsilon\} < C_4 e^{-\gamma t/\varepsilon} \quad (\gamma > 0),$$

while for $t > t_{2\varepsilon}$ one has $|u| < C_5\varepsilon$, as long as $y(t,\varepsilon)$ does not leave the domain D. The point $y(0,\varepsilon)$ lies strictly inside D, and the rate of change of $y(t,\varepsilon)$ is finite, so that $y(t,\varepsilon)$ can leave D only after a time $t_\varepsilon = O(1)$ (it is not excluded, of course, that $t_\varepsilon = \infty$). Put $t_\varepsilon^* = \min(t_\varepsilon, L)$. For $\varepsilon t_1 \leq t < t_\varepsilon^*$ we have, by what has been said,

$$|u(t,\varepsilon)| \leq C_4 e^{-\gamma t/\varepsilon} + C_5\varepsilon. \tag{11}$$

We shall prove later that, for sufficiently small ε, $t_\varepsilon^* = L$.

5. We need to estimate $|y(t,\varepsilon)-\bar{y}(t)|$ for $\varepsilon t_1 \le t \le t^*_\varepsilon$. It is more convenient to estimate on the interval $[\varepsilon t_1, t^*_\varepsilon]$ the quantity

$$\eta(t,\varepsilon) = y(t,\varepsilon) - \bar{y}(t) - I(t,\varepsilon),$$

where

$$I(t,\varepsilon) = \varepsilon T(y(t,\varepsilon)) \int_0^{\varphi(t,\varepsilon)} \{g[X(\theta,\bar{y}(t)),\bar{y}(t)] - \bar{g}[\bar{y}(t)]\}\, d\theta.$$

We want to prove that for $\varepsilon t_1 \leqslant t < t^*_\varepsilon$ inequality (6) holds; for this it is enough to show that, for the indicated values of t, $|\eta(t,\varepsilon)| = O(\varepsilon)$, $I(t,\varepsilon) = O(\varepsilon)$ uniformly in t. The second of these relations is obvious, since $g[X(\theta,\bar{y}(t)),\bar{y}(t)]$ is a periodic function of θ with period 1 and mean value $\bar{g}[\bar{y}(t)]$, so that the integral entering the expression for $I(t,\varepsilon)$ is $O(1)$. Let us now prove that $|\eta(t,\varepsilon)| = O(\varepsilon)$.

We have

$$\begin{aligned}\dot{\eta}(t,\varepsilon) &= g[X(\varphi(t,\varepsilon),y(t,\varepsilon)),y(t,\varepsilon)] + O(|u|+\varepsilon) - \bar{g}[\bar{y}(t)] \\ &\quad - \varepsilon T(y(t,\varepsilon))\dot{\varphi}(t,\varepsilon)\{g[X(\varphi(t,\varepsilon),\bar{y}(t)),\bar{y}(t)] - \bar{g}[\bar{y}(t)]\} \\ &\quad - \varepsilon T \int_0^{\varphi(t,\varepsilon)} \{\dot{g} - \dot{\bar{g}}\}\, d\theta - \varepsilon \dot{T}(y(t,\varepsilon)) \int_0^{\varphi(t,\varepsilon)} \{g - \bar{g}\}\, d\theta.\end{aligned}$$

The last two terms are, obviously, $O(\varepsilon)$. Further, the product

$$\varepsilon T(y(t,\varepsilon))\dot{\varphi}(t,\varepsilon) = 1 + O(\varepsilon + |u|)$$

and, consequently,

$$\dot{\eta}(t,\varepsilon) = g[X(\varphi(t,\varepsilon),y(t,\varepsilon)),y(t,\varepsilon)] - g[X(\varphi(t,\varepsilon),\bar{y}(t)),\bar{y}(t)] + O(\varepsilon + |u|).$$

Since $g[X(\varphi,y),y]$ depends smoothly on y and, as was proved, $y(t,\varepsilon) - \bar{y}(t) = \eta(t,\varepsilon) + O(\varepsilon)$, it follows that for $\varepsilon t_1 \leqslant t < t^*_\varepsilon$ we have

$$|\dot{\eta}(t,\varepsilon)| < B|\eta(t,\varepsilon)| + O(\varepsilon + |u|),$$

or, using (11),

$$|\dot{\eta}(t,\varepsilon)| < B|\eta(t,\varepsilon)| + C_6 e^{-\gamma t/\varepsilon} + C_7\varepsilon. \tag{12}$$

At the same time, evidently, $\eta(\varepsilon t_1,\varepsilon) = O(\varepsilon)$.

It is easy to show that from (12) there follows the estimate $|\eta(t,\varepsilon)| \leqslant \zeta(t,\varepsilon)$ for $t \in [\varepsilon t_1, t_\varepsilon^*)$, where

$$\dot{\zeta} = B\zeta + C_6 e^{-\gamma t/\varepsilon} + C_7\varepsilon, \qquad \zeta(\varepsilon t_1, \varepsilon) = \eta(\varepsilon t_1, \varepsilon).$$

Computing ζ from this, we obtain that $\zeta(t,\varepsilon) = O(\varepsilon)$ for $t \in [\varepsilon t_1, t_\varepsilon^*)$.

6. Inequalities (6), (6), obviously, follow for $t \in [\varepsilon t_1, t_\varepsilon^*)$ from the proved inequalities (6), (11) and from the system (10). It remains to show that $t_\varepsilon^* = L$. But this is obvious: if for $t \in [0, L]$ $\bar{y}(t)$ is distant from the boundary of the domain D by at least d ($d > 0$), and if it were the case that $t_\varepsilon^* = t_\varepsilon < L$, then for sufficiently small ε we would have $|y(t,\varepsilon) - \bar{y}(t)| < d/2$ for $t \in [\varepsilon t_1, t_\varepsilon)$, and we would obtain that $y(t_\varepsilon, \varepsilon)$ lies in D and is distant from the boundary D by at least $d/2$, and hence $y(t,\varepsilon)$ still does not leave D over the time t_ε. Thus the theorem is completely proved.

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REFERENCES

1. L. S. Pontryagin, *Proceedings of the Third All-Union Mathematical Congress*, 2, Moscow, 1956, p. 93; 3, Moscow, 1958, p. 570.

Note: Figure translations are in progress. See original paper for figures.