MATHEMATICS

I. B. SIMONENKO

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

I. B. SIMONENKO

BOUNDEDNESS OF SINGULAR INTEGRALS IN ORLICZ SPACES

(Presented by Academician S. L. Sobolev on 23 X 1959)

Let $\mathcal{L}_M^*(D)$ be the Orlicz space defined by a bounded measurable set D of m-dimensional space E_m and by the function

$$M(u) = \int_0^{|u|} p(t)\, dt,$$

where $p(t)$ is a nondecreasing function;

$$N(u) = \int_0^{|u|} q(t)\, dt,$$

where $q(t)$ is the right inverse of $p(t)$, and let $\mathcal{L}_N(D)$ be the set of functions V for which

$$\int_D N[v(P)]\, dP$$

exists.

As is known (([4], p. 83)), the space $\mathcal{L}_M^*(D)$ consists of functions u satisfying the condition

$$\int_D |u|\, |v|\, dP < \infty$$

for all $v \in \mathcal{L}_N(D)$; the norm in the space $\mathcal{L}_M^*(D)$ is introduced by the equality

$$\|u\|_M = \sup_v \int_D u(P) v(P)\, dP, \qquad \int_D N[v(P)]\, dP \leqslant 1.$$

Consider the singular integral

$$Kf = \int_D \frac{\Omega(P;\theta)}{|P-Q|^m} f(Q)\,dQ, \tag{1}$$

where

$$\theta = \frac{P-Q}{|P-Q|}$$

is a point of the unit sphere.

$\Omega(P;\theta)$ satisfies the following conditions:

1. $$\int_{S_1} \Omega(P;\theta)\,ds_\theta = 0,$$

   where S_1 is the unit sphere.
2. $\Omega(P;\theta)$ is continuous in θ for fixed P.
3. $$|\Omega(P;\theta_1) - \Omega(P;\theta_2)| \leqslant \omega(|\theta_1 - \theta_2|),$$

   where ω does not depend on P and satisfies condition 4.
4. $$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty.$$
5. $$|\Omega(P_1;\theta) - \Omega(P_2;\theta)| \leqslant B|P_1 - P_2|^\alpha; \quad \alpha > 0;$$

   B is a constant.

We subject the function $M(u)$ to the additional condition: for sufficiently large u the inequality

$$1 < \beta \leqslant \frac{up(u)}{M(u)} \leqslant \alpha \tag{2}$$

holds.

It is easy to prove that there exists a function M_1, equivalent to the function M (([4], p. 27)), satisfying inequality (2) for all u. At the same time ⋯

$\mathcal{L}^*_{M_1} = \mathcal{L}^*_M(D)$, and the norms $\|\ \|_{M_1}$ and $\|\ \|_M$ are equivalent (([4]), p. 130). Therefore, without loss of generality, we may assume that inequality (2) holds for all u.

Theorems on the boundedness of singular integrals in the class $\mathcal{L}_p$ $(p > 1)$ were obtained by S. G. Mikhlin ([1, 3]) and by A. Calderón and A. Zygmund ([2]). The space $\mathcal{L}_p$ is a special case of $\mathcal{L}_M^*(D)$, when $\alpha = \beta = p$. We have obtained a theorem generalizing the results of the authors mentioned.

Theorem. The singular operator (1) is defined and bounded in the space $\mathcal{L}_M^*(D)$, i.e.

$$\|Kf\|_M \leq C\|f\|_M, \tag{3}$$

where C depends only on $\alpha, \beta, \Omega, u_0, D$ *.

Proof. Introduce the notation:

$$K_\lambda(P;Q) = \begin{cases} \dfrac{\Omega(P;\theta)}{|P-Q|^m}, & \text{if } |P-Q| > \lambda, \\ 0, & \text{if } |P-Q| \leq \lambda; \end{cases}$$

$$|f(P)|_y = \begin{cases} f(P), & \text{if } f(P) \leq y, \\ y, & \text{if } f(P) > y; \end{cases}$$

$f^*(t)$ is the nonincreasing function equimeasurable ** with $f(Q)$, defined on the ray $0 \leq t < \infty$; $\beta_f(x) = \dfrac{1}{x}\int_0^x f^*(t)\,dt$; $\beta f(y)$ is the inverse function of $\beta_f(x)$; $\tilde{f}_\lambda = K_\lambda f =$

$$= \int_D K_\lambda(P;Q) f(Q)\,dQ; \quad E_y^\lambda \text{ is the set of points } P \in D, \text{ where } |\tilde{f}_\lambda(P)| \geq y.$$

Then for any $r > 1$ the following generalized inequality of A. Calderón and A. Zygmund ([2]) holds:

$$|E_y^\lambda| \leq \frac{C_1}{y^r}\int_D |f(P)|_y^r\,dP + C_1\beta f(y), \tag{4}$$

where C_1 is a constant depending only on D, r, and Ω.

This inequality was obtained in ([2]) for the case $r = 2$ and for a characteristic Ω depending on θ. The stated inequality is proved similarly, using the result of S. G. Mikhlin (([1]), p. 99, Theorem 3).

Since, for f satisfying the Hölder condition, $K_\lambda f$ tends uniformly to Kf as $\lambda \to 0$, the theorem will be proved if we verify that

$$\|\tilde{f}_\lambda\|_M \leq C\|f\|_M, \tag{5}$$

where C does not depend on λ. Obviously, it is sufficient to prove inequality (5) for $f \geq 0$, which we shall assume in what follows.

We give the necessary inequalities (see ([4]), pp. 39 and 251)

$$\frac{up(u)}{N[p(u)]} \geq \frac{\alpha}{\alpha - 1} \quad \text{for all } u; \tag{6}$$

$$\|u\|_M \leq 1 + \int_D M[\, u(P)\,]\, dP; \tag{7}$$

* When this note had been prepared for publication, we learned of the work ([6]), in which, under assumptions somewhat different from ours, boundedness is proved, but only in the case when $\Omega = \Omega(\theta)$.

** $f_1(t)$ $(0 \leq t < \infty)$ and $f(P)$ $(P \in D)$ are called equimeasurable if $mE(a \leq f_1 < b) = mE(a \leq f < b)$.

$$\int_D u(P) v(P)\, dP \leq \|u\|_M \|v\|_M; \tag{8}$$

if $\|u\|_M \leq 1$, then

$$\int_D M[u(P)]\, dP \leq \|u\|_M. \tag{9}$$

Let us proceed to estimate $\|\tilde{f}_\lambda\|_M$:

$$\|\tilde{f}_\lambda\|_M \leq 1 + \int_D M[\tilde{f}_\lambda(P)]\, dP = 1 + \int_0^\infty |E^\lambda|\, P(y)\, dy. \tag{10}$$

We estimate the last integral using inequality (4):

$$\int_0^\infty E_y^\lambda\, p(y)\, dy \leq C_1 \int_0^\infty \frac{p(y)}{y^r}\, dy \int_D |f(P)|_y^r\, dP + C_1 \int_0^\infty \beta_f(y) p(y)\, dy, \tag{11}$$

We consider the integrals on the right-hand side of (11) separately:

$$I_1 = \int_0^\infty \frac{p(y)}{y^r}\, dy \int_D |f(P)|_y^r\, dP = \int_D M[|f(P)|]\, dP + \int_D |f(P)|^r\, dP \int_{f(P)}^\infty \frac{p(y)}{y^r}\, dy.$$

Choosing $r > \alpha$ so that $\lim\limits_{y\to\infty} \dfrac{M(y)}{y^r} = 0$ ((4), p. 38), and integrating by parts, we obtain

$$\int_D |f(P)|^r \, dP \int_{f(P)}^{\infty} \frac{p(y)}{y^r} \, dy = -\int_D M[|f(P)|] \, dP + r \int_D |f(P)|^r \, dP \int_{f(P)}^{\infty} \frac{M(y)}{y^{r+1}} \, dy.$$

From the last equality and inequality (2) it follows that

$$\int_D |f(P)|^r \, dP \int_{f(P)}^{\infty} \frac{p(y)}{y^r} \, dy \leq \frac{\alpha}{r-\alpha} \int_D M[|f(P)|] \, dP.$$

Finally, for the integral I_1 we obtain the estimate

$$I_1 \leq \frac{r}{r-\alpha} \int_D M[|f(P)|] \, dP. \tag{12}$$

Let us turn to the second integral. Making the substitution $y = \beta_f(x)$, integrating by parts, and taking into account that

$$\lim_{x\to 0} xM[\beta_f(x)] \leq \lim_{x\to 0} \int_0^x M[|f^*(t)|] \, dt = 0,$$

we obtain a chain of inequalities:

$$\begin{aligned} I_2 &= \int_0^{\infty} \beta_f(y) p(y) \, dy = -\int_0^{\infty} x \, dM[|\beta_f(x)|] \\ &= -xM[|\beta_f(x)|]\Big|_0^{\infty} + \int_0^{\infty} M[|\beta_f(x)|] \, dx \leq \int_0^{\infty} M[|\beta_f(x)|] \, dx \\ &= xM[|\beta_f(x)|]\Big|_0^{\infty} - \int_0^{\infty} p[|\beta_f(x)|] f^*(x) \, dx + \int_0^{\infty} \beta_f(x) p[|\beta_f(x)|] \, dx. \end{aligned}$$

From the last equality and inequalities (2), (6), (7), (8) it follows that

$$(\beta - 1) \int_0^{\infty} M[\beta_f(x)] \, dx \leq \int_0^{\infty} p[\beta_f(x)] f^*(x) \, dx \leq \|f^*\|_M \|p[\beta_f(x)]\|_N \leq$$

$$\leq \left[1 + \int_0^{\infty} N\{p[\beta_f(x)]\} \, dx\right] \|f^*\|_M \leq \left[1 + \frac{\alpha - 1}{\alpha} \int_0^{\infty} \beta_f(x) p[\beta_f(x)] \, dx\right] \|f^*\|_M \leq$$

$$\leq 2\left[1+(\alpha-1)\int_0^\infty M[\beta_f(x)]\,dx\right]\|f\|_M^*.$$

Now it is easy to verify that on the sphere $\|f\|_M = k = \dfrac{\beta-1}{4(\alpha-1)} < 1$ the inequality

$$I_2 \leq \frac{4}{\beta-1}\,k \tag{13}$$

holds.

Substituting (13), (12) into (11), and then into (10), and using (9), we obtain on the sphere $\|f\|_M = k$ the estimate

$$\|\tilde{f}_\lambda\|_M \leq 1 + C_1 k\left(\frac{r}{r-\alpha}+\frac{4}{\beta-1}\right).$$

Finally, we shall have

$$\|K_\lambda\| \leq \frac{1}{k} + C_1\left(\frac{r}{r-\alpha}+\frac{4}{\beta-1}\right).$$

The theorem is proved.

Incidentally, from inequality (13) we have obtained the boundedness of the operator

$$\beta_f(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

in Orlicz spaces satisfying condition (2) for all u; its norm is

$$\|\beta\| \leq \frac{4\alpha}{\beta-1}.$$

For $\mathcal{L}_p$ $(p>1)$ this result was obtained by Hardy with a more precise estimate of the norm:

$$\|\beta\| \leq \frac{p}{p-1}$$

([5], p. 77).

The theorem proved admits a generalization to an unbounded domain D under the following additional conditions: 1) inequality (2) holds for all u; 2) Ω depends only on θ, or the conditions of S. G. Mikhlin' s theorem [3] are satisfied for some $p > \alpha$.

The theorem is applicable in estimating higher derivatives of elliptic equations, in the theory of one-dimensional singular equations, and in boundary-value problems for analytic functions.

The work was reported at the seminar on nonlinear problems of mechanics at Rostov-on-Don University.

The author expresses his gratitude to the seminar leader I. I. Vorovich, and also to its participants V. I. Yudovich and Yu. P. Krasovskii for useful discussion of the work.

Rostov-on-Don
State University

Received
22 X 1959

CITED LITERATURE

1. S. G. Mikhlin, *Uspekhi Mat. Nauk*, **3**, no. 3 (125) (1948).
2. A. Calderon, A. Zygmund, *Acta Math.*, **88**, no. 1–2, 85 (1952).
3. S. G. Mikhlin, DAN, **117**, no. 1, 28 (1957).
4. M. A. Krasnosel' skii, Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, 1953.
5. A. Zygmund, *Trigonometric Series*, 1939.
6. Koizumi Simuyu, *Proc. Japan Acad.*, **34**, no. 4 (1958).

---

* $\|f^*\|_M \le 2\|f\|_M$, since $\left\|\frac{f^*}{\|f\|_M}\right\| = \left\|\left(\frac{f}{\|f\|_M}\right)^*\right\| \le 1 + \int_D M\left(\frac{f}{\|f\|_M}\right) dP \le 2.$

Note: Figure translations are in progress. See original paper for figures.