MATHEMATICS

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1960

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Abstract

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ON A MULTIDIMENSIONAL INTEGRAL EQUATION ON A HALF-SPACE WITH KERNEL DEPENDING ON THE DIFFERENCE OF THE ARGUMENTS, AND ITS DISCRETE ANALOGUE

(Presented by Academician V. I. Smirnov on 16 XI 1959)

Let $t = (t_1, t_2, \dots, t_n)$ denote vectors of the n-dimensional real Euclidean space $E(= E_n)$, and let E^+ denote the half-space of E defined by the inequalities $0 \le t_1 < \infty,\ -\infty < t_j < \infty$ $(j = 2, 3, \dots, n)$.

Below we shall consider the multidimensional integral equation of the form

$$\varphi(t) - \int_{E^+} k(t-s)\varphi(s)\,ds = f(t) \qquad (t \in E^+), \tag{1}$$

where the function $k(t) \in L_1(E)$, while the given function $f(t)$ and the unknown function $\varphi(t)$ are assumed to belong to the space $D(E^+)$ (^{2}).

Separate special cases of the multidimensional equation of the form (1) occur in certain questions of mathematical physics (see, for example, (^{2})).

A detailed investigation of equation (1) and its discrete analogue in the one-dimensional case was carried out by M. G. Krein (^{1}).

The general theory of the multidimensional equation (1) is, in a certain sense, simpler than the general theory of the corresponding one-dimensional equation, since the index of equation (1) for $n > 1$, under the natural restrictions (4), is always equal to zero. At the same time, the method of effectively solving the one-dimensional equation (1) (the factorization method (^{1})) is entirely applicable also to the solution of the multidimensional equation (1). The difference that arises here between the one-dimensional and multidimensional equations recalls the difference between one-dimensional and multidimensional singular integral equations (^{3}).

In this note we also consider equation (6), which is the discrete analogue of equation (1).

As is known ([1]), in the one-dimensional case $n = 1$ the principal propositions concerning equation (6) essentially coincide with the corresponding propositions concerning equation (1). In contrast to this, for $n > 1$ the discrete analogue (6) of equation (1) differs essentially in its properties from equation (1). Thus, for example, when condition (7) is fulfilled, which is an analogue of condition (4), equation (6) may have both zero index and infinite index (see Theorems 2 and 3).

1. By $D(E^+)$ we shall denote one of the spaces of complex-valued functions

$$L_p(E^+)\,(p \geq 1), \qquad M(E^+), \qquad M_c(E^+), \qquad M_u(E^+), \qquad C(E^+), \qquad C_0(E^+). \tag{2}$$

The spaces listed are defined in the same way as in the one-dimensional case ([1], §6). If $k(t) \in L_1(E)$, then the operator K, defined by the equality

$$K\varphi = \int_{E^+} k(t-s)\varphi(s)\,ds \qquad (t \in E^+), \tag{3}$$

maps each of the spaces $D(E^+)$ into itself and is there a linear bounded operator, with $\|K\|_D \leqslant \|k(t)\|_{L_1}$.

Let $k(t) \in L_1(E)$ and

$$1 - \mathcal{K}(\lambda) = 1 - \int_E e^{i(\lambda,t)} k(t)\,dt \neq 0 \quad (\lambda \in E); \tag{4}$$

then, for $n > 1$, the function $\arg(1 - \mathcal{K}(\lambda))$ is a single-valued function and, by the Wiener–Lévy theorem,

$$\ln(1 - \mathcal{K}(\lambda)) = -\int_E e^{i(\lambda,t)} l(t)\,dt \qquad (l(t) \in L_1(E)).$$

Introducing the notation

$$G_+(\lambda) = \exp \int_{E^+} l(t) e^{i(\lambda,t)}\,dt, \qquad G_-(\lambda) = \exp \int_{E-E^+} l(t) e^{i(\lambda,t)}\,dt,$$

we obtain the factorization of the function $(1 - \mathcal{K}(\lambda))^{-1}$, needed below:

$$(1 - \mathcal{K}(\lambda))^{-1} = G_+(\lambda) G_-(\lambda) \qquad (\lambda \in E).$$

We note that the factors $G_\pm(\lambda)$ can be represented in the form

$$G_+(\lambda) = 1 + \int_{E^+} \gamma_1(t) e^{i(\lambda,t)}\,dt, \qquad G_-(\lambda) = 1 + \int_{E^+} \gamma_2(t) e^{-i(\lambda,t)}\,dt \quad (\gamma_1, \gamma_2 \in L_1(E^+)). \tag{5}$$

The functions $G_{\pm}^{-1}(\lambda)$ are reduced to the same form.

Using the method of proof of Theorem 7.1 from ([1]), it is easy to prove:

Theorem 1. *If the function $k(t) \in L_1(E)$ satisfies condition (4), then for any right-hand side $f \in D(E^+)$ equation (1) has a unique solution $\varphi(t) \in D(E^+)$, determined by the formula*

$$\varphi(t) = f(t) + \int_{E^+} \gamma(t; s) f(s)\, ds,$$

where

$$\gamma(t; s) = \gamma_1(t-s) + \gamma_2(s-t) + \int_{E^+} \gamma_1(t-r)\gamma_2(s-r)\, dr \qquad (s, r \in E^+),$$

and the functions $\gamma_1(t), \gamma_2(t)$ are determined from relations (5) and the equality

$$\gamma_1(t) = \gamma_2(t) = 0 \qquad (t \in E - E^+).$$

2. Denote by $j = (j_1, j_2, \dots, j_n)$ the points of the integer lattice $R(= R_n)$ of the space E, and by R^+ the subset of R defined by the inequalities

$$0 \leqslant j_1 < \infty, \qquad -\infty < j_m < \infty \quad (m = 2, 3, \dots, n).$$

By $l_p(R^+)$ $(p \geqslant 1)$ we denote the Banach space whose vectors are sets of complex numbers ξ_j $(j \in R^+)$ for which

$$\|\xi\|_{l_p}^p = \sum_{j \in R^+} |\xi_j|^p < +\infty.$$

By $m(R^+)$ we denote the Banach space of bounded sets $\xi = \{\xi_j\}_{j \in R^+}$, with the norm defined by

$$\|\xi\|_m = \sup_{j \in R^+} |\xi_j|.$$

Finally, by $c(R^+)$ $(c_0(R^+))$ we denote the subspace of $m(R^+)$ consisting respectively of all vectors $\xi = \{\xi_j\}_{j \in R^+}$ for which the limit $\lim_{j \to \infty} \xi_j$ exists (for which $\lim_{j \to \infty} \xi_j = 0$). In what follows, $d(R^+)$ will denote any of the spaces introduced.

The discrete analogue of equation (1) is naturally considered to be the infinite system of linear equations

$$\sum_{j \in R^+} a_{k-j} \xi_j = \eta_k \quad (k \in R^+), \tag{6}$$

where the vector $a = \{a_j\}_{j \in R}$ belongs to the space $d(R^+)$, and the given vector $\eta = \{\eta_k\}_{k \in R^+}$ and the unknown vector $\xi = \{\xi_j\}_{j \in R^+}$ are assumed to belong to the space $d(R^+)$. For simplicity, the subsequent exposition will be given for the case $n = 2$. Let $a = \{a_j\}_{j\in R} \in l_1(R)$ and let the function

$$A(\zeta) = A(\zeta_1, \zeta_2) = \sum_{k\in R} a_k \zeta_1^{k_1} \zeta_2^{k_2} \neq 0 \quad (|\zeta_1| = |\zeta_2| = 1). \tag{7}$$

The function $\arg A_1(\zeta)$, where

$$A_1(\zeta) = \zeta_1^{-\varkappa} \zeta_2^{-\nu} A(\zeta),$$

$$\varkappa = \frac{1}{2\pi} \left[\arg A(e^{i\varphi}, \zeta_2)\right]_{\varphi=0}^{\varphi=2\pi}, \qquad \nu = \frac{1}{2\pi} \left[\arg A(\zeta_1, e^{i\varphi})\right]_{\varphi=0}^{\varphi=2\pi},$$

is a single-valued function, and

$$\ln A_1(\zeta) = \sum_{j\in R} l_j \zeta_1^{j_1} \zeta_2^{j_2} \quad (\{l_j\}_{j\in R} \in l_1(R)).$$

Defining the functions $A_\pm(\zeta)$ by the equalities

$$A_+(\zeta) = \exp\left(\frac{1}{2}\sum_{j_2=-\infty}^{\infty} l_{0j_2}\zeta_2^{j_2} + \sum_{j_1=1}^{\infty}\sum_{j_2=-\infty}^{\infty} l_j \zeta_1^{j_1}\zeta_2^{j_2}\right),$$

$$A_-(\zeta) = \exp\left(\frac{1}{2}\sum_{j_2=-\infty}^{\infty} l_{0j_2}\zeta_2^{j_2} + \sum_{j_1=-\infty}^{-1}\sum_{j_2=-\infty}^{\infty} l_j \zeta_1^{j_1}\zeta_2^{j_2}\right) \quad (|\zeta_j| = 1),$$

we obtain the following factorization of the function $A^{-1}(\zeta)$:

$$A^{-1}(\zeta) = \zeta_1^{\varkappa} \zeta_2^{\nu} A_+(\zeta) A_-(\zeta) \quad (|\zeta_1| = |\zeta_2| = 1).$$

The factors $A_\pm(\zeta)$ can be represented in the form

$$A_\pm(\zeta) = \sum_{j\in R^+} \beta_j^\pm \zeta_1^{\pm j_1} \zeta_2^{j_2} \quad (|\zeta_1| = |\zeta_2| = 1),$$

where $\{\beta_j^+\}_{j\in R^+}$ and $\{\beta_j^-\}_{j\in R^+}$ belong to $l_1(R^+)$.

Theorem 2. If $a = \{a_j\}_{j\in R} \in l_1(R)$ and condition (7) is satisfied, then:

a) for $\varkappa \leq 0$, in all spaces $d(R^+)$ the homogeneous equation

$$\sum_{j\in R^+} a_{k-j}\xi_j = 0 \quad (k \in R^+) \tag{8}$$

has the unique zero solution;

b) for $\varkappa > 0$, equation (8) has an infinite set of linearly independent solutions; the general solution of equation (8) in all spaces $d(R^+)$ has the form

$$\xi_{j_1 j_2} = \sum_{p=0}^{\varkappa-1}\sum_{k=p}^{\infty} \beta_{j_1-p,\, j_2-k} c_k^{(p)} \quad (j \in R^+), \tag{9}$$

where $\{c_k^{(p)}\}_{k\in R_1}$ $(p = 0, 1, \dots, \varkappa - 1)$ are arbitrary vectors from $l_1(R_1)$.

Note that the function $x(\zeta)$, whose Fourier coefficients coincide with the numbers $\xi_{j_1 j_2}$, has the form

$$x(\zeta) = \sum_{j_1=0}^{\infty}\sum_{j_2=-\infty}^{\infty} \xi_{j_1 j_2}\zeta_1^{j_1}\zeta_2^{j_2} = A_+(\zeta_1, \zeta_2)\sum_{p=0}^{\varkappa-1} c_p(\zeta_2)\zeta_1^p,$$

where

$$c_p(\zeta_2) = \sum_{r=-\infty}^{\infty} c_r^{(p)}\zeta_2^r \qquad (p = 0, 1, \dots, \varkappa - 1).$$

Theorem 3. If $a = \{a_{ij}\}_{j\in R} \in l_1(R)$ and condition (7) is satisfied, then, in order that the system (6), where $\{\eta_j\}_{j\in R^+} \in d(R^+)$, have at least one solution $\{\xi_j\}_{j\in R^+} \in d(R^+)$, it is necessary and sufficient that the condition

$$\sum_{j\in R^+} \eta_j\chi_j = 0 \tag{10}$$

be satisfied for every solution $\{\chi_j\}_{j\in R^+} \in d(R^+)$ of the transposed system of homogeneous equations

$$\sum_{j\in R^+} a_{j-k}\chi_j = 0.$$

When condition (10) is satisfied, one of the solutions $\xi = \{\xi_j\}_{j\in R^+}$ of the system (6) is obtained by the formula

$$\xi_j = \sum_{k\in R^+} \gamma_{jk}\eta_k \quad (j \in R^+), \qquad \gamma_{jk} = \sum_{r\in R^+} \gamma_{j-r}^+\gamma_{k-r}^- \quad (j, k \in R^+),$$

and the numbers γ_j^+ and γ_{-j}^- are the Fourier coefficients, respectively, of the functions

$$\zeta_1^{(\varkappa+|x|)/2} A_+(\zeta), \qquad \zeta_1^{(\varkappa-|x|)/2} \zeta_2^y A_-(\zeta) \qquad (|\zeta_j| = 1).$$

According to Theorem 2, condition (10) need be observed only in the case $\varkappa < 0$. In the latter case, (10) may be replaced by the following equivalent condition

$$\sum_{j_1=0}^{\infty} \sum_{k=-\infty}^{\infty} \eta_{j_1,k} \chi_{j_1,j_2-k}^{(p)} = 0 \qquad (j_2 = 0, \pm 1, \dots;\ p = 0, 1, \dots, \varkappa - 1),$$

where the numbers $\chi_j^{(p)}$ $(j \in R^+)$ are the Fourier coefficients of the function $\zeta_1^p A_-(\zeta_1^{-1}, \zeta_2^{-1})$ $(p = 0, 1, \dots, \varkappa - 1)$.

3. The theorems given above for equations (1) and (6) admit a generalization to the case of the paired integral equation[4]

$$\varphi(t) - \int_E k_1(t-s)\varphi(s)\,ds = f(t) \qquad (t \in E^- = E - E^+),$$

$$\varphi(t) - \int_E k_2(t-s)\varphi(s)\,ds = f(t) \qquad (t \in E^+),$$

its transposed equation

$$\varphi(t) - \int_{E^-} k_1(s-t)\varphi(s)\,ds - \int_{E^+} k_2(s-t)\varphi(s)\,ds = f(t) \qquad (t \in E)$$

and their discrete analogues.[4]

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Received
13 XI 1959

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Note: Figure translations are in progress. See original paper for figures.