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1960

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Abstract

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A New Integral Representation of a Piecewise-Analytic Function and Its Application

(Presented by Academician V. I. Smirnov on 28 VI 1960)

Consider the following boundary-value problem of the Riemann-problem type. A simple smooth closed contour L is given, dividing the plane into two domains D^+, D^-. Determine two functions $\Phi^+(z)$, $\Phi^-(z)$ ($\Phi^-(\infty) = 0$), analytic respectively in the domains D^+, D^-, and satisfying on the contour the boundary condition

$$\sum_{k=0}^{n}\left[a_k(t)\frac{d^k\Phi^+(t)}{dt^k} + \int_L A_k(t,\tau)\frac{d^k\Phi^+(\tau)}{d\tau^k}\,d\tau\right] - \sum_{k=0}^{p}\left[b_k(t)\frac{d^k\Phi^-(t)}{dt^k} + \int_L B_k(t,\tau)\frac{d^k\Phi^-(\tau)}{d\tau^k}\,d\tau\right] = f(t), \tag{1}$$

where $a_k(t)$, $b_k(t)$, $f(t)$ are given continuous functions, with $a_n(t)$, $b_p(t)$ satisfying a Hölder condition (condition H) and not vanishing; $A_k(t,\tau)$, $B_k(t,\tau)$ are Fredholm kernels.

This problem was first considered by L. G. Magnaradze in [1]. A simpler investigation was later carried out by Yu. M. Krikunov [2, 3], who, with the aid of an integral representation of the required functions, reduced the problem to a singular integral equation with Cauchy kernel.

In the present paper a new integral representation is indicated for the required functions in the boundary-value problem (1), which makes it possible to reduce it directly to a Fredholm-type integral equation. This introduces considerable simplifications into the solution, since the equation obtained by Yu. M. Krikunov requires a regularization involving cumbersome calculations.

We begin with the formulation of auxiliary propositions.

Lemma 1. Let $c(t)$ satisfy condition H on L, be different from zero, and let $\operatorname{ind} c(t) = \varkappa \geq 0$; let $f^+(z)$ be analytic in D^+ and continuous in the closed domain $\overline{D^+}$; let $f^-(z)$ be analytic in D^- and have a zero of order $p+1$ at infinity. Under these conditions the representation holds

$$f^+(z) = \frac{1}{2\pi i}\int_L \frac{\nu(\tau)}{c(\tau)}\,\frac{d\tau}{\tau - z}, \qquad z \in D^+; \tag{2a}$$

$$f^-(z) = -\frac{1}{z^p\cdot 2\pi i}\int_L \nu(\tau)\,\frac{d\tau}{\tau - z}, \qquad z \in D^-, \tag{2b}$$

where $\nu(\tau)$ is a complex function of points of the contour L, satisfying condition H, determined from the given functions $f^+(z)$ and $f^-(z)$ up to $\varkappa$ arbitrary constants.

Lemma 2. Suppose that, under the conditions of the preceding lemma, $\operatorname{ind} c(t) = \varkappa < 0$. Then the representation holds

$$f^+(z) = \frac{1}{2\pi i} \int_L \frac{v(\tau)}{c(\tau)} \frac{d\tau}{\tau - z} + p_0 + p_1 z + \dots + p_{-\varkappa-1} z^{-\varkappa-1}, \tag{3a}$$

$$f^-(z) = -\frac{1}{z^p \cdot 2\pi i} \int_L v(\tau) \frac{d\tau}{\tau - z}, \tag{3b}$$

where p_k are constants determined by the functions $f^+(z)$ and $f^-(z)$.

Applying Lemmas 1 and 2 to the functions $\Phi^{+(n)}(z)$ and $\Phi^{-(p)}(z)$, one can prove the following theorems.

Theorem 1. Suppose $c(t)$ satisfies condition H and $\operatorname{ind} c(t) = \varkappa$. Then, for $n > \varkappa$, the integral representation holds

$$\Phi^+(z) = \frac{(-1)^n}{(n-1)!} \frac{1}{2\pi i} \int_L \frac{v(\tau)}{c(\tau)} (\tau - z)^{n-1} \ln\left(1 - \frac{z}{\tau}\right) d\tau + \sum_{k=1}^{n-\varkappa} \frac{c_k}{(k-1)!} z^{k-1}, \tag{4a}$$

$$\Phi^-(z) = \frac{(-1)^p}{(p-1)!} \frac{1}{2\pi i} \int_L \frac{v(\tau)}{\tau^p} (\tau - z)^{p-1} \ln\left(1 - \frac{\tau}{z}\right) d\tau + \sum_{k=1}^{p-1} d_{k-1} z^{k-1}, \tag{4b}$$

where $v(\tau)$ is a function satisfying condition H; c_k are complex constants, with $v(\tau)$ and c_k determined by $\Phi^+(z)$ and $\Phi^-(z)$ uniquely, while the constants d_k have the form

$$d_0 = \frac{(-1)^p}{(p-1)!} \frac{\beta_0}{2\pi i} \int_L v(\tau) \tau^{-1}\, d\tau + \Phi^-(\infty),$$

$$d_k = \frac{(-1)^p}{(p-1)!} \frac{\beta_k}{2\pi i} \int_L v(\tau) \tau^{-(k+1)}\, d\tau, \qquad k = 1, 2, \dots, p-2; \tag{5}$$

here

$$\beta_k = \sum_{q=k+1}^{p-1} \frac{(-1)^q}{q-k} c_{p-1}^q.$$

Theorem 2. If, under the conditions of the preceding theorem, $n \leqslant \varkappa$, then the integral representation holds

$$\Phi^{+}(z) = \frac{(-1)^{n}}{(n-1)!}\frac{1}{2\pi i}\int_{L}\frac{v(\tau)}{c(\tau)}(\tau - z)^{n-1}\ln\left(1-\frac{z}{\tau}\right)d\tau + c_{1}, \tag{6a}$$

$$\Phi^{-}(z) = \frac{(-1)^{p}}{(p-1)!}\frac{1}{2\pi i}\int_{L}\frac{v(\tau)}{\tau^{p}}(\tau - z)^{p-1}\ln\left(1-\frac{\tau}{z}\right)d\tau + \sum_{k=1}^{p-1} d_{k-1}z^{k-1}, \tag{6b}$$

where $c_1 = \Phi^{+}(0)$; d_k are determined by formulas (5).

We now proceed to reducing the boundary-value problem (1) to an integral equation. For this purpose we set $c(t) = a_n(t)/b_p(t)t^p$ and use the integral representations (4a), (4b) or (6a), (6b), depending on whether $n > \varkappa$ or $n \leqslant \varkappa$. By means of differentiation we can determine all derivatives of the sought functions $\Phi^{+}(z)$ and $\Phi^{-}(z)$ entering into the boundary condition (1). The derivatives of order n of $\Phi^{+}(z)$ and of order p of $\Phi^{-}(z)$ will then be represented by Cauchy-type integrals. Passing to the limit as $z \to t$ $(t \in L)$ and using the Sokhotski formulas

for the limiting values of the Cauchy-type integral, we obtain the integral equation

$$\frac{a_n(t) + t^{-p}b_p(t)}{2}\nu(t) + \frac{1}{2\pi i}\int_{L} a_n(t)\left[\frac{b_p(\tau)}{a_n(\tau)\tau^{p}} - \frac{b_p(t)}{a_n(t)t^{p}}\right]\frac{\nu(\tau)}{\tau - t}\,d\tau + \tag{7}$$

$$+\int_{L} K(t,\tau)\nu(\tau)\,d\tau + \sum_{k=1}^{n-\varkappa} h_k g_k(t) = f(t),$$

where h_k are arbitrary constants; $g_k(t)$ are known functions; $K(t,\tau)$ is a Fredholm kernel. For $n \leqslant \varkappa$, only the first term should be retained from the sum. Since $a_n(t)$ and $b_p(t)$ satisfy condition H, the kernel entering the first integral term will also be a Fredholm kernel. Thus problem (1) has been reduced to a Fredholm integral equation.

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Received
24 VI 1960

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.