



Soviet-era science, translated into English

S. Ya. Khavinson

1960

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196001.96634>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

S. Ya. Khavinson

On Approximation on Sets of Analytic Capacity Zero

(Presented by Academician V. I. Smirnov, 17 XI 1959)

Let G be an arbitrary domain with boundary Γ , and let $\infty \in G$. We shall use the definitions and notation of our notes ^(1,2).

§ 1. From the dual expressions for analytic capacity, with the aid of the results of ^(1,2), there follow the following theorems concerning the approximation of functions given in a neighborhood of the set Γ .

Theorem 1. *If Γ has zero analytic capacity ($\Omega(\Gamma) = 0$), then for every $q \geq 1$ we have: for an arbitrary $\varepsilon > 0$ there is a neighborhood d of the set Γ , $d \supset \Gamma$, such that for every finitely connected domain $D \subset G$, whose boundary Γ_D is rectifiable and lies in d , the inequality*

$$\inf_{\substack{\varphi \in E_q(D) \\ \varphi(\infty)=0, \varphi'(\infty)=0}} \int_{\Gamma_D} |1 + \varphi(\zeta)|^q ds < \varepsilon \quad (1)$$

is satisfied.

Conversely, if for some $q = q_0 \geq 1$ inequality (1) holds for every $\varepsilon > 0$, then $\Omega(\Gamma) = 0$.

Remark. The assertion of the theorem for $q = 1$ and $q = 2$ is essentially contained in Garabedian ⁽³⁾, and for $q = 1$ also in Ahlfors ⁽⁴⁾.

Corollary. If the approximation property described in the theorem holds for some $q = q_0 \geq 1$, then it holds for all $q \geq 1$.

The assertion of Theorem 1 can now be supplemented by the following results.

Theorem 2. *Let $\Omega(\Gamma) = 0$, and let the function $F(z)$ be analytic on Γ . Let $q = 1$ or $q = 2$.*

Then for any sequence of finitely connected domains $\{G_n\}$, exhausting G , with rectifiable boundaries Γ_n , we have

$$\lim_{n \rightarrow \infty} \inf_{\substack{\varphi \in E_q(G_n) \\ \varphi(\infty)=0}} \int_{\Gamma_n} |F(\zeta) + \varphi(\zeta)|^q ds = 0. \quad (2)$$

Theorem 3. Let $\Omega(\Gamma) = 0$ and $l(\Gamma) < +\infty$ ($l(\Gamma)$ is the Painlevé length of the set Γ), and let the function $F(z)$ be continuous in some neighborhood of Γ . For every $q \geq 1$, equality (2) holds. Here $\{G_n\}$ is the same as in Theorem 2, and, in addition, $\overline{\lim} \partial l \Gamma_n < +\infty$.

We outline the proof of Theorem 2. By virtue of the duality relations ⁽⁵⁾ we have:

$$\inf_{\substack{\varphi \in E_q(G_n) \\ \varphi(\infty)=0}} \left\{ \int_{\Gamma_n} |F(\zeta) + \varphi(\zeta)|^q ds \right\}^{1/q} = \sup_{\substack{f \in E_p^1(G_n) \\ f(\infty)=0}} \left| \int_{\Gamma_n} f(\zeta) F(\zeta) d\zeta \right|, \quad (3)$$

$$\frac{1}{p} + \frac{1}{q} = 1;$$

in the case $q = 1$, $E_\infty^1(G_n) = B^1(G_n)$. Let $q = 2$ and suppose that (2) is not satisfied. Choosing from the sequence of functions $\{f_n^*(z)\}$, extremal for equality (3), a subsequence converging uniformly in G to a function $f^*(z) \in E_2^1(\{G_n\})$, we shall have

$$\int_\gamma f^*(\zeta) F(\zeta) d\zeta \neq 0$$

(γ is any contour enclosing Γ , on which $F(z)$ is still analytic), and hence $f^*(z) \neq 0$. This means the nontriviality of the class $E_2^1(\{G_n\})$, which, by virtue of the results from ⁽¹⁾, contradicts the condition $\Omega(\Gamma) = 0$. The case $q = 1$ is still simpler.

For arbitrary $q \geq 1$ this argument fails at the point where we refer to the nontriviality of the class $E_p(\{G_n\})$, since in note ⁽¹⁾ we were able to establish the triviality of $E_p(\{G_n\})$, $p \geq 1$, under the condition $\Omega(\Gamma) = 0$ only with the additional condition $l(\Gamma) < +\infty$. However, in this latter case, instead of Theorem 2, the stronger assertion of Theorem 3 holds. To prove the latter, one must approximate on Γ the function $F(z)$ by an analytic function $\Phi(z)$ (this is possible, since Γ is everywhere discontinuous and, consequently, M. A. Lavrent'ev's theorem ⁽⁶⁾ is applicable), and then argue as above.

All the theorems just given may be formulated somewhat differently, for example:

Theorem 1'. In order that $\Omega(\Gamma) = 0$, it is necessary that for every $q \geq 1$, and sufficient that for some $q = q_0 \geq 1$, the following property hold: for every $\varepsilon > 0$ one can find a neighborhood $d \subset G$ of Γ such that, for every domain $D \subset G$ with boundary $\Gamma_D \subset d$, there exists a rational fraction $R_D(z)$ with poles on Γ , $R_D(\infty) = 0$, $R_D'(\infty) = 0$, for which

$$\int_{\Gamma_D} |1 + R_D(\zeta)|^q ds < \varepsilon.$$

§ 2. The investigation of extremal problems with additional conditions has led us to a new approach to the concept of analytic capacity, which we shall now set forth.

Lemma. Let Γ be a compact set; $\varphi_1(\zeta), \dots, \varphi_n(\zeta), \Phi(\zeta)$ be continuous functions on Γ (complex-valued); B_K be the set of complex measures $\mu(e)$ defined on the Borel subsets of Γ and satisfying the conditions

$$\int_{\Gamma} |d\mu| \leq K; \quad \left| \int_{\Gamma} \varphi_{\nu}(\zeta) d\mu \right| \leq \varepsilon_{\nu}, \quad \nu = 1, \dots, n; \quad (4)$$

$K, \varepsilon_1, \dots, \varepsilon_n$ are given nonnegative numbers. Then

$$\max_{\mu \in B_K} \left| \int_{\Gamma} \Phi(\zeta) d\mu \right| = \min_{\lambda_1, \dots, \lambda_n} \left[K \max_{\zeta \in \Gamma} \left| \Phi(\zeta) + \sum_{\nu=1}^n \lambda_{\nu} \varphi_{\nu}(\zeta) \right| + \sum_{\nu=1}^n \varepsilon_{\nu} |\lambda_{\nu}| \right]. \quad (5)$$

The proof of this lemma is obtained with the aid of the so-called duality lemmas of functional analysis.

Theorem 4. Let the function $F(z) \in E_1(\{G_n\})$ ($\{G_n\}$ as in Theorems 2 and 3); $\Phi(z)$ be analytic on Γ .

Then, for all sufficiently large $K > 0$, the equality

$$\begin{aligned} & \sup_{\substack{f \in B^1(G) \\ f(\infty)=0}} \left| \int_{\Gamma} f(\zeta) F(\zeta) \Phi(\zeta) d\zeta \right| = \\ & = \inf_{\substack{\{a_{\nu}\} \subset G \\ \{\lambda_{\nu}\}}} \left[K \max_{\zeta \in \Gamma} \left| \Phi(\zeta) + \sum_{\nu=1}^n \frac{\lambda_{\nu}}{\zeta - a_{\nu}} \right| + \sum_{\nu=1}^n |\lambda_{\nu} F(a_{\nu})| \right]. \end{aligned} \quad (6)$$

Taking, in particular, in formula (6) $\Phi(z) \equiv 1$, we obtain: for any $F(z) \in E_1(\{G_n\})$, $F(\infty) = 1$, and sufficiently large $K > 0$,

$$\Omega(\Gamma) = \frac{1}{2\pi} \inf_{\substack{\{a_{\nu}\} \subset G \\ \{\lambda_{\nu}\}}} \left[K \max_{\zeta \in \Gamma} \left| 1 + \sum_{\nu=1}^n \frac{\lambda_{\nu}}{\zeta - a_{\nu}} \right| + \sum_{\nu=1}^n |\lambda_{\nu} F(a_{\nu})| \right]. \quad (7)$$

If $l(\Gamma) < +\infty$, then in the preceding formulas (6) and (7) one may take $F(z) \equiv 1$.

We shall not give the proof here in full; we indicate only that it is based on the preceding lemma and on the possibility, for every $f(z) \in B^1(G)$, $f(\infty) = 0$, of the representation

$$f(z)F(z) = \int_{\Gamma} \frac{d\mu}{\zeta - z}, \quad (8)$$

where $\mu(e)$ is a certain measure given on Γ , and moreover $\int_{\Gamma} |d\mu| \leq K$, with K depending only on $F(z)$. Representation (8) is proved with the aid of Radon's theorem on sequences of measures. From Theorem 4 one can obtain:

Theorem 5. In order that $\Omega(\Gamma) = 0$, it is necessary and, in the case $l(\Gamma) < +\infty$, sufficient that for every $\varepsilon > 0$ there exist points $a_1, \dots, a_n \in G$ and numbers $\lambda_1, \dots, \lambda_n$ such that

$$\max_{\zeta \in \Gamma} \left| 1 + \sum_1^n \frac{\lambda_\nu}{\zeta - a_\nu} \right| < \varepsilon, \quad \sum_1^n |\lambda_\nu| < \varepsilon. \quad (9)$$

Proof. If $\Omega(\Gamma) = 0$, then $L(z) \equiv 1$ (2). But always $L(z) \in E_1(\{G_n\})$, and from (7) with $F(z) \equiv L(z) \equiv 1$ follows (9). If (9) holds and $l(\Gamma) < +\infty$, then from (7) with $F(z) \equiv 1$ it follows that $\Omega(\Gamma) = 0$.

Using the theorem of M. A. Lavrent'ev (6) and formula (6), one obtains

Theorem 6. If $\Omega(\Gamma) = 0$ and $\Psi(z)$ is continuous on Γ , then for every $\varepsilon > 0$ there exist points $a_1, \dots, a_n \in G$ and numbers $\lambda_1, \dots, \lambda_n$ for which

$$\max_{\zeta \in \Gamma} \left| \Psi(\zeta) + \sum_1^n \frac{\lambda_\nu}{\zeta - a_\nu} \right| < \varepsilon, \quad \sum_1^n |\lambda_\nu| < \varepsilon.$$

Remark. Results differing from those presented in § 2 only in the form of exposition were found (from somewhat different considerations) independently of me and almost simultaneously with me also by V. P. Khavin. I learned of this from the kind communications of V. P. Khavin.

Moscow Civil Engineering Institute
named after V. V. Kuibyshev

Received
11 XI 1959

References

1. S. Ya. Khavinson, DAN, **128**, No. 5 (1959).
2. S. Ya. Khavinson, DAN, **128**, No. 6 (1959).
3. P. R. Garabedian, Trans. Am. Math. Soc., **67**, No. 1, 1 (1949).
4. L. V. Ahlfors, Comm. Math. Helv., **24** (1950).
5. S. Ya. Khavinson, Matem. sborn., **36** (78), No. 3, 445 (1955).

6. M. A. Lavrent' ev, Tr. Fiz.-matem. inst. im. V. A. Steklova, **5**, 159 (1934).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.