ON THE PHENOMENON OF EJECTION BY AN ELECTRIC DISCHARGE

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Abstract

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ON THE PHENOMENON OF EJECTION BY AN ELECTRIC DISCHARGE

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A theory is considered of an electric discharge in a medium with finite conductivity. It is shown that an axisymmetric discharge is in principle accompanied by motion of the medium, exerting an ejecting action on it. Similarity laws for an axisymmetric discharge are derived, and self-similar solutions are given for its equations.

§ 1. The equations of magnetic hydrodynamics for the case of a plane discharge in an incompressible medium have the form

$$(\mathbf{v}\nabla)H = \nu_m \Delta H; \qquad \operatorname{div} \mathbf{v} = 0; \qquad (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p_m, \tag{1}$$

where H is the magnetic-field intensity (in the present case there is only a component along the z-axis, if the x, y plane is the plane in which the current-density vector $\mathbf{j}$ lies); ν_m is the magnetic viscosity; ρ is the density of the medium; p is the pressure; $\mathbf{v}$ is the velocity vector; $p_m = p + H^2/8\pi$.

If there was no motion of the medium before the discharge, then in the process of a stationary discharge there will in many cases be no motion either, and the intensity H, as in the usual electrodynamic case, satisfies the equation $\Delta H = 0$, while the pressure is found from the relation $p + H^2/8\pi = \text{const}$.

§ 2. The equations of magnetic hydrodynamics for the case of a stationary axisymmetric discharge in an incompressible medium have the form

$$u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} - \frac{Hv}{y} = \nu_m \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{1}{y}\frac{\partial H}{\partial y} - \frac{H}{y^2}\right); \qquad \frac{\partial yu}{\partial x} + \frac{\partial yv}{\partial y} = 0;$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_m}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x^2}\right); \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p_m}{\partial y} - \frac{H^2}{4\pi\rho y} + \nu\left(\frac{\partial^2 v}{\partial y^2} + \frac{1}{y}\frac{\partial v}{\partial y} - \frac{v}{y^2} + \frac{\partial^2 v}{\partial x^2}\right);$$

x is the axis of symmetry, ν is the coefficient of viscosity; in the present case only the component of the magnetic-field intensity vector perpendicular to the meridional plane x, y is nonzero.

Let us show that an axisymmetric discharge in a medium at rest is impossible. Indeed, supposing the contrary and differentiating the third equation of system (2) with respect to y, and the fourth with respect to x, and subtracting the results, we obtain that $\partial H/\partial x = 0$, i.e. the magnetic-field intensity satisfies in this case the equation

$$\frac{\partial^2 H}{\partial y^2} + \frac{1}{y}\frac{\partial H}{\partial y} - \frac{H}{y^2} = 0. \tag{3}$$

The only component of the current-density vector that is nonzero in the case under consideration is

$$j_x = \frac{c}{4\pi}\left(\frac{\partial H}{\partial y} + \frac{H}{y}\right). \tag{4}$$

Differentiating equation (4) with respect to y, we obtain

$$\frac{\partial^2 H}{\partial y^2} + \frac{1}{y}\frac{\partial H}{\partial y} - \frac{H}{y^2} = \frac{4\pi}{c}\frac{dj_x}{dy}, \tag{5}$$

Comparing equations (5) and (3), we arrive at the conclusion that for a finite discharge (i.e., a discharge with current density varying along y) they are incompatible.

Thus, the axisymmetric discharge under consideration is always accompanied by motion of the medium. In light of this, the inconsistency of the propositions of magnetohydrostatics for the axisymmetric discharge being studied becomes clear: the solutions obtained there cannot be realized even in media with arbitrarily large, but finite, conductivity σ.

§ 3. In what follows we shall consider the case when the dimensionless combination $R = I/\nu_m c\sqrt{\rho}$ is large, i.e., $R \gg 1$ (c is the speed of light; I is the total current in the discharge).

The quantity R is the ratio of the characteristic magnitude of the inertial terms to the characteristic magnitude of the dissipative terms in the first equation of system (2). In the case when R is large, a boundary-layer type phenomenon occurs: the discharge is concentrated in the vicinity of the x-axis; the magnetic-field strength is of order $I/c\delta$ (δ is the thickness of the discharge column); the velocity u is of order $I/c\sqrt{\rho}\,\delta$, i.e., it is a large quantity; on the basis of the first equation of system (2), the thickness of the discharge column δ is of order L/R.

On the basis of the continuity equation, the order of the velocity v will be $I/c\sqrt{\rho}\,L$, i.e., $v \ll u$. Estimating the terms entering into the system of equations

(2), we find that, for the phenomenon under consideration, equations (2), with relative accuracy up to quantities of order $(\delta/L)^2$, take the form

$$\frac{1}{y}\left(\frac{\partial\psi}{\partial y}\frac{\partial H}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial H}{\partial y}\right)+\frac{H}{y^2}\frac{\partial\psi}{\partial x}=\nu_m\left(\frac{\partial^2 H}{\partial y^2}+\frac{1}{y}\frac{\partial H}{\partial y}-\frac{H}{y^2}\right);$$
$$\frac{1}{y}\left[\frac{1}{y}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial\psi}{\partial y}\right)\right]=-\frac{1}{\rho}\frac{\partial p_m}{\partial x}+\nu\left(\frac{1}{y}\frac{\partial^3\psi}{\partial y^3}-\frac{1}{y^2}\frac{\partial^2\psi}{\partial y^2}+\frac{1}{y^3}\frac{\partial\psi}{\partial y}\right);$$
$$\frac{\partial p_m}{\partial y}=-\frac{H^2}{4\pi y}\left(u=\frac{1}{y}\frac{\partial\psi}{\partial y};\quad v=-\frac{1}{y}\frac{\partial\psi}{\partial x}\right);$$
$$\frac{1}{y}\rho c_p\left(\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial y^2}+\frac{1}{y}\frac{\partial T}{\partial y}\right)+\nu\rho\frac{1}{y^2}\left(\frac{\partial^2\psi}{\partial y^2}-\frac{1}{y}\frac{\partial\psi}{\partial y}\right)^2+\frac{\nu_m}{4\pi}\frac{1}{y^2}\left(\frac{\partial yH}{\partial y}\right)^2 \qquad (6)$$

(c_p is the specific heat at constant pressure; T is the absolute temperature; k is the coefficient of thermal conductivity).

The last equation is the heat-balance equation. Let us consider the case of the discharge shown in Fig. 3. Integrating the third equation of system (6) with respect to y from y to ∞, we have

$$p_m = p_{m\infty} + \int_y^\infty \frac{H^2}{4\pi y}\, dy.$$

Differentiating the last relation with respect to x and noting that, for the type of discharge considered, $\partial|H|/\partial x < 0$, on the basis of the second equation of system (6) we obtain ($\nu = 0$)

$$du/dt > 0,$$

i.e., everywhere $u > 0$, if $u|_{x=0} = 0$. Thus, the expanding axisymmetric discharge exerts an ejecting action on the medium.

The proof given evidently remains valid also for a compressible medium with variable conductivity.

§ 4. Let us carry out, in the equations of system (6), a similarity transformation of the form $\varphi_1 = \mu_\varphi \varphi_2$, where μ_φ is a dimensionless constant quantity (by the quantity φ we mean all the variables and constants entering equations (6)). We shall require that the equations for the quantities φ_2 be the same as for the quantities φ_1; then we obtain the following system of equations satisfied by the quantities μ_φ:

$$\mu_\psi = \mu_{\nu_m}\mu_x; \qquad \mu_\psi = \mu_\nu \mu_x; \qquad \mu_\psi = \frac{\mu_k}{\mu_\rho \mu_{c_p}}\,\mu_x;$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

$$\mu_{p_m} = \mu_H^2 = \frac{\mu_\rho}{\mu_\nu^4}\mu_\psi^2; \qquad \mu_T = \frac{1}{\mu_\rho \mu_{c_p}}\mu_H^2. \tag{7}$$

The quantities μ_u and μ_σ are determined from the formulas:

$$\mu_u = \frac{\mu_\psi}{\mu_y^2}; \qquad \mu_x = \frac{\mu_\psi}{\mu_x \mu_y}. \tag{8}$$

In what follows we shall assume that p_m in the discharge is much greater than the pressure at infinitely large distances from it (i.e. $p_\infty = 0$) and that the medium is at rest at infinity.

Fig. 1

Fig. 2

Consider two discharges in two different media, where the quantities $\omega = \nu/\nu_m$ and the magnetic Prandtl number $\mathrm{Pr}_m = \dfrac{\rho c_p \nu_m}{k}$ are the same; then the second and third equations of system (7) are equivalent to the first. Let, further, discharges between two point electrodes A and B (see Fig. 1), located on the x-axis, be considered, and let the ratio of the distances between the electrodes in discharge 1 and in discharge 2 be equal to a. The only nonhomogeneous boundary condition will be: on the segment $0 < x < x_B$, $\lim_{y\to\infty} yH = 2I/c$, where I is the total current in the discharge.

Putting $\mu_x = \mu_y = a$, we obtain

$$\mu_{p_m} = \mu_H^2 = \frac{\mu_\rho \mu_{\nu_m}^2}{a^2}; \qquad \mu_u = \mu_\nu = \frac{\mu_{\nu_m}}{a}; \qquad \mu_T = \frac{\mu_{\nu_m}^2}{\mu_{c_p} a^2}. \tag{9}$$

In order that the two discharges under consideration be similar in accordance with rule (9), it is necessary that the current magnitudes be related as

$$\mu_I = \sqrt{\mu_\rho \mu_{\nu_m}}. \tag{10}$$

Let us further consider two discharges in media with identical ω and Pr_m between a point electrode A and an infinitely thin ring of small radius B (see Fig.

2), with equal distances from point A to the ring ($\mu_x = 1$) and with different radii of the rings, related as b ($\mu_y = b$). The similarity law for this type of discharge will be the following:

$$\mu_u = \frac{\mu_{\nu_m}}{b^2}; \qquad \mu_\sigma = \frac{\mu_{\nu_m}}{b}; \qquad \mu_H^2 = \mu_{p_m} = \frac{\mu_\rho \mu_{\nu_m}^2}{b^4};$$

$$\mu_I = \frac{\sqrt{\mu_\rho}\,\mu_{\nu_m}}{b}; \qquad \mu_T = \frac{\mu_{\nu_m}^2}{\mu_{c_p} b^4}. \tag{11}$$

§ 5. The system of equations (6) for an axisymmetric discharge admits the following class of self-similar solutions

$$H = \frac{IR^{1+\gamma}}{c}\, y^\gamma h(\zeta); \quad \psi = v_m x f(\zeta); \quad p_m = \frac{I^2 R^2}{c^2}\, x^\delta g(\zeta); \tag{12}$$

$$T = \frac{I^4}{c^4 \rho v_m k}\, x^q t(\zeta); \quad \zeta = Ryx^\alpha,$$

where the functions $h(\zeta)$, $f(\zeta)$, $g(\zeta)$, $t(\zeta)$ satisfy the system of equations

$$\begin{gathered}
\gamma f h + \alpha\gamma\zeta f' h + \zeta f h' + (\gamma^2 - 1)h + (2\gamma + 1)\zeta h' + \zeta^2 h'' - h(f + \alpha\zeta f') = 0; \\
(1+\alpha){f'}^2 - f f'' + \frac{f f'}{\zeta} + \alpha {f'}^2 = -(\delta g + \alpha\zeta g')\zeta^2 + \omega\left(\zeta f''' - f'' + \frac{f'}{\zeta}\right); \\
g' = -\frac{\zeta^{2\gamma-1}}{4\pi}\, h^2; \\
\mathrm{Pr}_m(q f' t - f t') = \zeta t'' + t' + \frac{\zeta}{4\pi}\left[(\gamma+1)h + \zeta h'\right]^2 + \omega\left(\frac{f''}{\zeta} - \frac{f'}{\zeta^2}\right)^2.
\end{gathered} \tag{13}$$

The constants $\gamma, \alpha, \delta, q$ are related by

$$\delta = 2 + 4\alpha; \quad \gamma = -\frac{1}{\alpha} - 2;$$

$$q = -2\gamma\alpha. \tag{14}$$

§ 6. Suppose that the distance between the electrodes A and B is large (Fig. 1), and study the behavior of the discharge in the neighborhood of electrode A. The solution sought will be self-similar with the following values of the constants:

Fig. 3

Fig. 3

Figure 3: Fig. 3

$$\alpha = \gamma = -1; \quad \delta = q = -2; \quad \zeta = \frac{Ry}{x}. \tag{15}$$

The value of the constant γ has been chosen so that the boundary condition $\lim_{y\to\infty} yH = 2I/c$ is satisfied.

Thus, in order to solve the problem posed, it is necessary to investigate the system of equations:

$$\begin{gathered}
-fh + \zeta(fh)' + \zeta^2 h'' - \zeta h' - h(f - \zeta f') = 0; \\
-ff'' + \frac{ff'}{\zeta} - f'^2 = (2g + \zeta g')\zeta^2 + \omega\left(\zeta f''' - f'' + \frac{f'}{\zeta}\right); \\
4\pi\zeta^3 g' = -h^2; \quad \mathrm{Pr}_m(2f't + ft') + \zeta t'' + t' + \frac{\zeta^3}{4\pi}h'^2 + \omega\left(\frac{f''}{\zeta} - \frac{f'}{\zeta^2}\right)^2 = 0.
\end{gathered} \tag{16}$$

In conclusion, let us note the existence of a self-similar solution, analogous to solution (16), for a compressible perfect viscous gas in the case of an arbitrary parameter R and constant coefficients of viscosity, thermal conductivity, and magnetic viscosity; this solution has the form

$$H = \frac{1}{y}h(\zeta); \quad \psi = xf(\zeta); \quad p_m = \frac{1}{x^2}g(\zeta); \quad \rho = l(\zeta); \quad T = \frac{1}{x^2}t(\zeta); \quad \zeta = \frac{Ry}{x}.$$

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Note: Figure translations are in progress. See original paper for figures.